Reconstruction Theorems in Category Theory - Exercise Sheet 1

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Exercise 1.

(i) Let A be a ring, M and M' right A-modules and N and N' left A-modules. Let further $f\colon M \to M'$ be a morphism of right A-modules and $g\colon N \to N'$ be a morphism of left A-modules. The map $M \times N \to M' \otimes_A N', (m, n) \mapsto f(m) \otimes g(n)$ induces a morphism $f \otimes g\colon M \otimes_A N \to M' \otimes_A N'$. Show that

$$f \otimes g = (f \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes g) = (\mathrm{id}_{M'} \otimes g) \circ (f \otimes \mathrm{id}_N).$$

(ii) Let R, S and T be rings, M and M' be (R, S)-bimodules, $f\colon M \to M'$ a morphism of (R, S)-bimodules, N and N' be (S, T)-bimodules and g a morphism of (S, T)-bimodules. Show that the morphism $f \otimes g$ constructed in (i) is a morphism of (R, T)-bimodules.

(iii) Let A be a commutative ring and let M and N be symmetric (A, A)-bimodules. Show that there exists a unique morphism of (A, A)-bimodules $\tau_{M,N}\colon M \otimes_A N \to N \otimes_A M$ such that $\tau_{M,N}(m \otimes n) = n \otimes m$ for all $m \in M$ and $n \in N$.

(iv) Let A be a commutative ring and M, M', N and N be symmetric (A, A)-bimodules. Let $f\colon M \to M'$ and $g\colon N \to N'$ be morphisms of (A, A)-bimodules. Show that

$$(g \otimes f) \circ \tau_{M,N} = \tau_{M',N'} \circ (f \otimes g) = (g \otimes \mathrm{id}_{M'}) \circ \tau_{M',N} \circ (f \otimes \mathrm{id}_N) = (\mathrm{id}_{N'} \otimes f) \circ \tau_{M,N'} \circ (\mathrm{id}_M \otimes g).$$

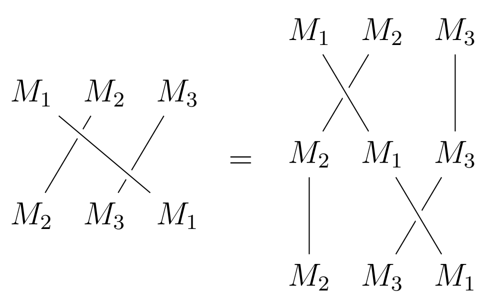
(v) A be a commutative ring and let M_1, M_2 and M_3 be symmetric (A, A)-bimodules. Show that

$$\tau_{M_1, M_2 \otimes_A M_3} = (\mathrm{id}_{M_2} \otimes \tau_{M_1,M_3}) \circ (\tau_{M_1,M_2} \otimes \mathrm{id}_{M_3}).$$

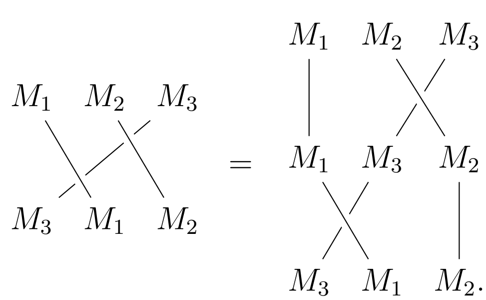
and

$$\tau_{M_1 \otimes_A M_2, M_3} = (\tau_{M_1,M_3} \otimes \mathrm{id}_{M_2}) \circ (\mathrm{id}_{M_1} \otimes \tau_{M_2,M_3}).$$

Note: These identities can be visualized by the string diagrams



and



Exercise 2.

(i) Let A be a ring. Show that a left A-module M is isomorphic to $A \otimes_A M$ and that a right A-module M is isomorphic to $M \otimes_A A$.

(ii) Let k be a commutative ring and let $(A, \mathrm{m}_A, \eta_A)$ and $(A', \mathrm{m}_{A'}, \eta_{A'})$ be k-algebras. Show that $A \otimes_k A'$ becomes a k-algebra with respect to the multiplication

$$A \otimes_k A' \otimes_k A \otimes_k A' \xrightarrow{\mathrm{id}_A \otimes \tau_{A',A} \otimes \mathrm{id}_{A'}} A \otimes_k A \otimes_k A' \otimes_k A' \xrightarrow{\mathrm{m}_A \otimes \mathrm{m}_{A'}} A \otimes_k A'$$

and unit $\eta_A \otimes \eta_{A'} \colon k \to A \otimes_k A'$, where we identify $k \otimes_k k$ with k using (i).

Exercise 3.

Let k be a commutative ring, A_1 and A_2 be k-algebras and M be a k-module. We say that a left A_1-module structure $\psi_1 \colon A_1 \otimes_k M$ and a left A_2-module structure $\psi_2 \colon A_2 \otimes_k M \to M$ on M commute if the diagram

$$\begin{array}{ccccc} A_1 \otimes_k A_2 \otimes_k M & \xrightarrow{\mathrm{id}_{A_1} \otimes \psi_2} & A_1 \otimes_k M & & \\ \Big\downarrow{\scriptstyle \tau_{A_1,A_2} \otimes \mathrm{id}_M} & & & \searrow^{\psi_1} & \\ & & & & M \\ & & & \nearrow_{\psi_2} & \\ A_2 \otimes_k A_1 \otimes_k M & \xrightarrow{\mathrm{id}_{A_2} \otimes \psi_1} & A_2 \otimes_k M & & \end{array} \tag{1}$$

commutes.

(i) Show that a left $A_1 \otimes_k A_2$-module structure on M, given by a morphism $\psi \colon (A_1 \otimes_k A_2) \otimes_k M \to M$, induces commuting left A_i-module structures $\psi_i \colon A_i \otimes_k M \to M$ $(i = 1, 2)$ on M.

(ii) Show that conversely commuting left A_1-module and left A_2-module structures on M, given by morphisms $\psi_1 \colon A_1 \otimes_k M \to M$ and $\psi_2 \colon A_2 \otimes_k M \to M$, induce a left $A_1 \otimes_k A_2$-module structure $\psi \colon (A_1 \otimes_k A_2) \otimes_k M \to M$ on M.

Exercise 4.

Let k be a commutative ring. Let further $f \colon M \to M'$ and $g \colon N \to N'$ be morphisms of k-modules.

(i) Show that $f \otimes g \colon M \otimes_k N \to M' \otimes_k N'$ is surjective if f and g are surjective.

(ii) Give an example of injective morphisms f and g such that $f \otimes g$ is not injective.