Reconstruction Theorems in Category Theory - Exercise Sheet 2

University of Siegen
Dr. Florian Heiderich

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Exercise 1.
Let R, S and T be rings, $(E_i)_{i\in I}$ be a family of (R,S)-bimodules and $(F_j)_{j\in J}$ be a family of (S,T)-bimodules.

(i) Show that there is a morphism of (R,T)-bimodules

$$f\colon (\prod_{i\in I} E_i)\otimes_S (\prod_{j\in J} F_j) \to \prod_{(i,j)\in I\times J} (E_i \otimes_S F_j).$$

such that $f((e_i)_{i\in I}\otimes (f_j)_{j\in J}) = (e_i\otimes f_j)_{(i,j)\in I\times J}$.

(ii) Show that there is an isomorphism of (R,T)-bimodules

$$g\colon (\bigoplus_{i\in I} E_i)\otimes_S (\bigoplus_{j\in J} F_j) \to \bigoplus_{(i,j)\in I\times J} (E_i \otimes_S F_j).$$

such that $g((\sum_{i\in I} e_i)\otimes(\sum_{j\in J} f_j)) = \sum_{(i,j)\in I\times J}(e_i\otimes f_j)$.

(iii) Let k be a commutative ring, E a k-module and F a free k-module with basis $(b_j)_{j\in J}$. Show that $E\otimes_k F$ is isomorphic to $E^{(J)}$ as k-module and that every element of $E\otimes_k F$ can be uniquely written as $\sum_{j\in J} e_j\otimes b_j$, where $(e_j)_{j\in J}$ is a family of elements in E, only finitely many of them not being equal to 0.

(iv) Let k be a commutative ring and let E and F be free k-modules with bases $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$, respectively. Show that $E\otimes_k F$ is a free k-module with basis $(a_i\otimes b_j)_{(i,j)\in I\times J}$.

(v) (*) Show that the morphism f in (i) is in general neither injective nor surjective.

Exercise 2.
Let $C := \mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ and $D := \mathbb{Z}\oplus 2\mathbb{Z}/4\mathbb{Z}$.

(i) Show that C is a $\mathbb{Z}$-coalgebra with respect to the comultiplication Δ and counit ε given by the morphisms of $\mathbb{Z}$-modules $\Delta\colon C\to C\otimes_{\mathbb{Z}} C$ and $\varepsilon\colon C\to\mathbb{Z}$ defined by

$$\Delta((1,0)) := (1,0)\otimes(1,0) \qquad \Delta((0,1)) := (1,0)\otimes(0,1) + (0,1)\otimes(1,0)$$
$$\varepsilon((1,0)) := 1 \qquad \varepsilon((0,1)) := 0.$$

(ii) Show that D becomes a $\mathbb{Z}$-coalgebra with respect to comultiplication Δ_1 and counit ε_1 given by the morphisms of $\mathbb{Z}$-modules $\Delta_1\colon D\to D\otimes_{\mathbb{Z}} D$ and $\varepsilon_1\colon D\to\mathbb{Z}$ defined by

$$\Delta_1((1,0)) := (1,0)\otimes(1,0) \qquad \Delta_1((0,2)) := (1,0)\otimes(0,2) + (0,2)\otimes(1,0)$$
$$\varepsilon_1((1,0)) := 1 \qquad \varepsilon_1((0,2)) := 0.$$

(iii) Show that D becomes a $\mathbb{Z}$-coalgebra with respect to comultiplication Δ_2 and counit ε_2 given by the morphisms of $\mathbb{Z}$-modules $\Delta_2\colon D\to D\otimes_{\mathbb{Z}} D$ and $\varepsilon_2\colon D\to\mathbb{Z}$ defined by

$$\Delta_2((1,0)) := (1,0)\otimes(1,0) \qquad \Delta_2((0,2)) := (1,0)\otimes(0,2) + (0,2)\otimes(0,2) + (0,2)\otimes(1,0)$$
$$\varepsilon_2((1,0)) := 1 \qquad \varepsilon_2((0,2)) := 0.$$

(iv) Show that the natural inclusion of D in C is a morphism of $\mathbb{Z}$-coalgebras from $(D, \Delta_1, \varepsilon_1)$ to (C, Δ, ε) and also from $(D, \Delta_2, \varepsilon_2)$ to (C, Δ, ε).

(v) Show that $(D, \Delta_1, \varepsilon_1)$ and $(D, \Delta_2, \varepsilon_2)$ are not isomorphic as $\mathbb{Z}$-coalgebras.

Exercise 3.
Let k be a field and M be a monoid.

(i) Show that the morphism

$$ {}_k\mathcal{M}(kM, k) \otimes_k {}_k\mathcal{M}(kM, k) \to {}_k\mathcal{M}(kM \otimes_k kM, k), \quad f \otimes g \mapsto (c \otimes c' \mapsto f(c)g(c')) \quad (1) $$

is an isomorphism if and only if M is finite.

(ii) Show that there is an isomorphism of k-algebras

$$ {}_k\mathcal{M}(kM, k) \cong k^M, $$

where k^M is the k-algebra of functions on M, kM is the grouplike coalgebra with basis M and ${}_k\mathcal{M}(kM, k)$ is the convolution algebra. (The algebra structure on ${}_k\mathcal{M}(kM, k)$ is given by

$$ {}_k\mathcal{M}(kM, k) \otimes {}_k\mathcal{M}(kM, k) \xrightarrow{(1)} {}_k\mathcal{M}(kM \otimes_k kM, k) \xrightarrow{{}_k\mathcal{M}(\Delta, k)} {}_k\mathcal{M}(kM, k) $$

as multiplication and

$$ k \xrightarrow{\sim} {}_k\mathcal{M}(k, k) \xrightarrow{{}_k\mathcal{M}(\varepsilon, k)} {}_k\mathcal{M}(kM, k) $$

as unit.)

(iii) Let M be finite.

(a) Show that ${}_k\mathcal{M}(kM, k)$ becomes a k-coalgebra with comultiplication given by the composition of

$$ {}_k\mathcal{M}(kM, k) \xrightarrow{{}_k\mathcal{M}(\mathrm{m}, k)} {}_k\mathcal{M}(kM \otimes_k kM, k), $$

with the inverse of (1) and with counit given by

$$ {}_k\mathcal{M}(kM, k) \xrightarrow{{}_k\mathcal{M}(\eta, k)} {}_k\mathcal{M}(k, k) \xrightarrow{\sim} k. $$

(b) Show that ${}_k\mathcal{M}(kM, k)$ (with the coalgebra structure from (iii)a) is isomorphic to k^M as k-coalgebra.

(c) Show that M is a group if and only if ${}_k\mathcal{M}(kM, k) \cong k^M$ is a k-Hopf algebra.

(d) Show that M is commutative if and only if k^M is cocommutative.

Exercise 4.
Let k be a field and H and H' be k-Hopf algebras with antipodes S and S', respectively. Then every morphism $f \colon H \to H'$ of k-bialgebras preserves the antipodes, i.e. we have

$$ S' \circ f = f \circ S. $$

Exercise 5.
Let k be a commutative ring and $(B, \mathrm{m}, \eta, \Delta, \varepsilon)$ be a bialgebra. Show that an antipode for B is unique if it exists.