Reconstruction Theorems in Category Theory - Exercise Sheet 3

University of Siegen — Fall Term 2016/2017
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Exercise 1.
Let k be a field, C be a k-coalgebra and V be a k-module. Let further $(e_i)_{i\in I}$ be a k-basis of V.

(i) Let $\rho\colon V \to V \otimes_k C$ be a morphism of k-modules and let $c_{ij} \in C$ be defined by

$$\rho(e_j) = \sum_{i\in I} e_i \otimes c_{ij}.$$

Then ρ is a right C-comodule structure on V if and only if

$$\Delta(c_{ij}) = \sum_{k\in I} c_{ik} \otimes c_{kj} \qquad \text{and} \qquad \varepsilon(c_{ij}) = \delta_{ij}$$

for all $i, j \in I$.

(ii) Let $\lambda\colon V \to C \otimes_k V$ be a morphism of k-modules and let $c_{ij} \in C$ be defined by

$$\lambda(e_i) = \sum_{j\in I}^{n} c_{ij} \otimes e_j.$$

Then λ is a left C-comodule structure on V if and only if

$$\Delta(c_{ij}) = \sum_{k\in I} c_{ik} \otimes c_{kj} \qquad \text{and} \qquad \varepsilon(c_{ij}) = \delta_{ij}.$$

for all $i, j \in I$.

Exercise 2.
Let k be a field. Let further V be a finite dimensional k-module and C be a k-coalgebra. Then under the isomorphisms

$${}_k\mathcal{M}(V, V \otimes_k C) \cong {}_k\mathcal{M}(V \otimes_k {}^*V, C) \cong {}_k\mathcal{M}({}^*V, C \otimes_k {}^*V)$$

right C-comodule structures $\rho\colon V \to V \otimes_k C$ on V are in 1-1 correspondence with left C-comodule structures $\lambda\colon {}^*V \to C \otimes_k {}^*V$ on ${}^*V$.

Exercise 3.
Let k be a commutative ring and A be a k-bialgebra. Then the category of left A-modules is monoidal with the tensor product of two left A-modules M and N defined as $M \otimes_k N$ with the left A-module structure given by

$$a(m \otimes n) := \sum_{(a)} (a_{(1)}m) \otimes (a_{(2)}n). \tag{1}$$

and unit object k with A-module structure induced by $a\lambda := \varepsilon(a)\lambda$ for all $a \in A$ and $\lambda \in k$. Denote this monoidal category by ${}_{A/k}\mathcal{M}$. The forgetful functor $\omega\colon {}_{A/k}\mathcal{M} \to {}_k\mathcal{M}$ to the category of left k-modules ${}_k\mathcal{M}$ is a monoidal functor.

Exercise 4.

(i) For any monoidal category $(\mathsf{C}, \otimes, I)$ there is a monoidal category $(\mathsf{C}^{rev}, \otimes^{rev}, I)$, called the ***reverse*** of C. The underlying category C^{rev} is C and the tensor product of two objects $X \otimes^{rev} Y$ is defined as the tensor product $Y \otimes X$ in C. The associativity constraint

$$a^{rev}_{X,Y,Z} \colon (X \otimes^{rev} Y) \otimes^{rev} Z \to X \otimes^{rev} (Y \otimes^{rev} Z)$$

in $(\mathsf{C}^{rev}, \otimes^{rev}, I)$ is defined as the inverse

$$a^{-1}_{Z,Y,X} \colon Z \otimes (Y \otimes X) \to (Z \otimes Y) \otimes X$$

of the associativity constraint $a_{Z,Y,X}$ of $(\mathsf{C}, \otimes, I)$. The left unit constraint $l^{rev}_X \colon I \otimes^{rev} X \to X$ in C^{rev} is defined as the right unit constraint $r_X \colon X \otimes I \to X$ in C and the right unit constraint $r^{rev}_X \colon X \otimes^{rev} I \to X$ in C^{rev} as the left unit constraint $l_X \colon I \otimes X \to X$ in C.

(ii) If $(\mathsf{C}, \otimes, I)$ is braided, then the identity functor $F \coloneqq \mathrm{id}_{\mathsf{C}} \colon \mathsf{C} \to \mathsf{C}^{rev}$ becomes a monoidal functor with respect to $\phi_{X,Y} \coloneqq \Psi_{Y,X}$ for all objects X and Y of C.

$$\begin{array}{ccc} F(X) \otimes^{rev} F(Y) & \xrightarrow{\phi_{X,Y}} & F(X \otimes Y) \\ \| & & \| \\ Y \otimes X & \xrightarrow{\Psi_{Y,X}} & X \otimes Y \end{array}$$

and the isomorphism

$$\phi_0 \colon I \to I$$

being the identity on I. Moreover the category C^{rev} is braided with respect to

$$\Psi_{X,Y}{}^{rev} \colon X \otimes^{rev} Y \to Y \otimes^{rev} X \tag{2}$$

being the unique isomorphism that makes the diagram

$$\begin{array}{ccc} X \otimes^{rev} Y & \xrightarrow{\Psi_{X,Y}{}^{rev}} & Y \otimes^{rev} X \\ \| & & \| \\ Y \otimes X & \xrightarrow{\Psi_{Y,X}} & X \otimes Y \end{array}$$

commutative. Then the monoidal functor $F = \mathrm{id} \colon \mathsf{C} \to \mathsf{C}^{rev}$ is a braided monoidal functor.