Reconstruction Theorems in Category Theory - Exercise Sheet 4

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Fall Term 2016/2017
Due on December 15, 2016

Exercise 1.

Let k be a field and M be a monoid. Consider the category of *M-graded vector spaces* that has as objects the k-vector spaces V with decomposition $V = \oplus_{m\in M} V_m$ and as morphisms the k-linear morphisms that preserve the grading.

(i) Show that this category is monoidal, the tensor product of $V = \oplus_m V_m$ and $W = \oplus_m W_m$ being defined as $V \otimes_k W = \oplus_{m\in M}(\oplus_{m_1 m_2 = m} V_{m_1} \otimes_k W_{m_2})$ and unit object $I := k$, which is M-graded via $I_1 = k$ and $I_m = 0$ for all $1 \neq m \in M$.

(ii) Show that this monoidal category is monoidally equivalent to the monoidal category $\mathcal{M}^{kM}$ of right kM-comodules over the monoid bialgebra kM.

Exercise 2.

Let $\mathbb{C}C_n$ be the group algebra of the cyclic group C_n of order n over the complex numbers $\mathbb{C}$ and let g be a generator for C_n. The $\mathbb{C}$-algebra $\mathbb{C}C_n$ becomes a $\mathbb{C}$-bialgebra with respect to comultiplication Δ and counit ε defined by $\Delta(g^a) = g^a \otimes g^a$ and $\varepsilon(g^a) = 1$ for all $a = 0, \dots, n-1$.

(i) Show that the monoidal category ${}_{\mathbb{C}C_n/\mathbb{C}}\mathcal{M}$ of left $\mathbb{C}C_n$-modules over $\mathbb{C}$ is equivalent to the monoidal category of C_n-graded $\mathbb{C}$-vector spaces (see exercise 1).

Hint: A left $\mathbb{C}C_n$-module V is a C_n-graded $\mathbb{C}$-vector space $V = \oplus_{a=0}^{n-1} V_a$ with $V_a := \{v \in V \mid gv = e^{\frac{2\pi i a}{n}} v\}$ for $a = 0, \dots, n-1$.

(ii) Show that besides $R = 1 \otimes 1$ also $R = n^{-1} \sum_{a,b=0}^{n-1} e^{\frac{-2\pi i ab}{n}} g^a \otimes g^b$ is a quasitriangular structure on $\mathbb{C}C_n$.

(iii) Show that the braiding corresponding to $R = n^{-1} \sum_{a,b=0}^{n-1} e^{\frac{-2\pi i ab}{n}} g^a \otimes g^b$ on the monoidal category ${}_{\mathbb{C}C_n/\mathbb{C}}\mathcal{M}$ of left $\mathbb{C}C_n$-modules over $\mathbb{C}$ is given by $\Psi_{V,W}(v \otimes w) = e^{\frac{2\pi i |v||w|}{n}} w \otimes v$ for homogeneous elements $v \in V$ and $w \in W$ of degree $|v|$ and $|w|$, respectively.

(iv) Show that the induced braided monoidal category is symmetric if and only if $n = 1$ or $n = 2$. [1]

Exercise 3.

Let k be a commutative ring and let (A, R) be a quasitriangular bialgebra over k. Show that $(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1$ and that $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Exercise 4.

Let k be a field and $(A, \mathcal{R})$ be a dual quasitriangular bialgebra over k. Show that $\mathcal{R}(a \otimes 1) = \varepsilon(a) = \mathcal{R}(1 \otimes a)$ for all $a \in A$ and that

$$\sum_{(a),(b),(c)} \mathcal{R}(a_{(1)} \otimes b_{(1)}) \mathcal{R}(a_{(2)} \otimes c_{(1)}) \mathcal{R}(b_{(2)} \otimes c_{(2)}) = \sum_{(a),(b),(c)} \mathcal{R}(b_{(1)} \otimes c_{(1)}) \mathcal{R}(a_{(1)} \otimes c_{(2)}) \mathcal{R}(a_{(2)} \otimes b_{(2)}).$$

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[1] In case $n = 1$ this is the symmetric monoidal category of $\mathbb{C}$-vector spaces with the usual twist morphism. In case $n = 2$ this is the symmetric monoidal category of supervector spaces.