# Reconstruction Theorems in Category Theory - Exercise Sheet 5

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**Exercise 1.**
Let $(\mathsf{C}, \otimes, I)$ be a strict monoidal category and let $\varepsilon\colon A \otimes B \to I$ be a pairing between objects $A$ and $B$ of $\mathsf{C}$. Show that $\varepsilon$ is exact if and only if

$$\mathsf{C}(X, Y \otimes A) \to \mathsf{C}(X \otimes B, Y), g \mapsto (\mathrm{id}_Y \otimes \varepsilon) \circ (g \otimes \mathrm{id}_B) =: f_g.$$

is a bijection for all objects $X$ and $Y$ of $\mathsf{C}$, i.e. the functor $- \otimes B\colon \mathsf{C} \to \mathsf{C}$ is left adjoint to $- \otimes A\colon \mathsf{C} \to \mathsf{C}$.

**Exercise 2.**
Let $k$ be a field and $G$ be a group. Show that the monoidal category ${}_{kG/k}\mathfrak{m}$ of left $kG$-modules of finite dimension over $k$ is left rigid.

**Exercise 3.**
Let $k$ be a field and $(A, \mathrm{m}_A, \eta_A)$ be a $k$-algebra. Let $\mathsf{C}$ be the category with objects the pairs $(V, \rho_V)$ consisting of a finite dimensional $k$-vector space $V$ and a morphism $\rho_V\colon V \to V \otimes_k A$ and morphisms from $(V, \rho_V)$ to $(W, \rho_W)$ the homomorphisms of $k$-vector spaces $f\colon V \to W$ such that $\rho_W \circ f = (f \otimes \mathrm{id}_A) \circ \rho_V$.

(i) Show that $\mathsf{C}$ is monoidal with the tensor product of $(V, \rho_V)$ and $(W, \rho_W)$ defined as $V \otimes_k W$ together with $V \otimes_k W \to V \otimes_k W \otimes_k A$ given by the composition

$$V \otimes_k W \xrightarrow{\rho_V \otimes \rho_W} V \otimes_k A \otimes_k W \otimes_k A \xrightarrow{\mathrm{id}_V \otimes \tau \otimes \mathrm{id}_W} V \otimes_k W \otimes_k A \otimes_k A \xrightarrow{\mathrm{id}_V \otimes \mathrm{id}_W \otimes \mathrm{m}_A} V \otimes_k W \otimes_k A$$

and unit $(k, \eta_A)$.

(ii) Show that the functor $F\colon \mathsf{C} \to {}_k\mathcal{M}, (V, \rho_V) \mapsto V$ is monoidal.

(iii) Let $(V, \rho_V)$ be an object of $\mathsf{C}$, $\{e_1, \dots, e_n\}$ be a $k$-basis of $V$, $\rho_V(e_i) = \sum_j e_j \otimes a_{ji}$, $(W, \rho_W)$ be another object, $\{f_1, \dots f_n\}$ be a $k$-basis of $W$ and $\rho_W(f_i) = \sum_j f_j \otimes b_{ji}$ such that the morphisms of $k$-vector spaces $\varepsilon\colon V \otimes_k W \to k, e_i \otimes f_j \mapsto \delta_{i,j}$ and $\eta\colon k \to W \otimes_k V$ make $W$ into a right dual of $V$ in the category of $k$-vector spaces.

Show that with $\alpha := (a_{ij})_{i,j}$ and $\beta := (b_{ij})_{i,j}$

(a) $(W, \rho_W)$ is a right dual of $(V, \rho_V)$ if and only if $({}^t\alpha)\beta = 1 = \beta({}^t\alpha)$, i.e. if and only if $\beta = ({}^t\alpha)^{-1}$ and

(b) $(W, \rho_W)$ is a left dual of $(V, \rho_V)$ if and only if $({}^t\beta)\alpha = 1 = \alpha({}^t\beta)$, i.e. if and only if $\beta = {}^t(\alpha^{-1})$.

where ${}^t\alpha$ and ${}^t\beta$ denote the transposes of the matrices $\alpha$ and $\beta$.

(iv) Define $\alpha^{(0)} := \alpha$, $\alpha^{(n+1)} := {}^t((\alpha^{(n)})^{-1})$ for $n \geq 0$ and $\alpha^{(n-1)} := ({}^t(\alpha^{(n)}))^{-1}$ for $n \leq 0$ (if they exist). Call $\alpha$ *totally invertible* if $\alpha^{(n)}$ exists for all $n \in \mathbb{Z}$. Show that the full subcategory of $C$ consisting of those $(V, \rho_V)$ with $\rho_V(e_i) = \sum_j e_j \otimes a_{ji}$ with $\alpha = (a_{ij})_{i,j}$ totally invertible is autonomous.

(v) Show that left duals and right duals in $\mathsf{C}$ coincide if $A$ is commutative.

(vi) Give an example of a $k$-algebra $A$ and an object $(V, \rho_V)$ in $\mathsf{C}$ that has a left dual but no right dual.