Reconstruction Theorems in Category Theory - Exercise Sheet 6

University of Siegen	Fall Term $2016/2017$
Dr. Florian Heiderich	Due on January 26, 2017

Exercise 1.

Let k be a field and H be a k-Hopf algebra and H' be the k-bialgebra obtained from H with the same k-coalgebra structure but with respect to the k-algebra structure given by multiplication m' := m $\circ \tau$, where τ : $H \otimes_k H \to H \otimes_k H$ is defined by $\tau(a \otimes b) = b \otimes a$, and the unit of H. Show that H' has an antipode S' if and only if the antipode S of H is bijective. In this case $S' = S^{-1}$.

Exercise 2.

Let k be a field and H be a Hopf algebra over k. Let $_{H/k}\mathfrak{m}$ be the monoidal category of left H-modules of finite dimension over k.

- (i) Show that $_{H/k}\mathfrak{m}$ is left rigid.
- (ii) Show that $_{H/k}\mathfrak{m}$ is right rigid if the antipode S of H is invertible (with respect to the composition).

Exercise 3.

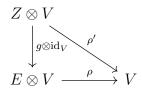
Let k be a field, H be a k-Hopf algebra and $_{H/k}\mathcal{M}$ be the monoidal category of left H-modules with tensor product \otimes_k .

- (i) Show that $_{H/k}\mathcal{M}$ is left closed.
- (ii) Show that $_{H/k}\mathcal{M}$ is right closed if the antipode of H is invertible (with respect to the composition).

Exercise 4.

Let k be a field and $_k\mathcal{M}$ be the category of k-vector spaces.

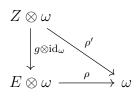
(i) Let V be a k-vector space. Show that there exists a k-vector space E together with a morphism $\rho: E \otimes V \to V$ such that for every k-vector space Z and every morphism $\rho': Z \otimes V \to V$ of k-vector spaces there exists a unique morphism $g: Z \to E$ of k-vector spaces such that the diagram



commutes. Show that E and ρ are determined up to unique isomorphism.

- (ii) Let E and $\rho: E \otimes V \to V$ be as in (i). Show that E has a k-algebra structure such that V becomes a left E-module with respect to ρ .
- (iii) Let C be a small category and $\omega: C \to {}_k\mathcal{M}$ be a functor. Show that there exists a k-vector space E and a natural transformation $\rho: E \otimes \omega \to \omega$ such that for any k-vector

space Z and any natural transformation $\rho': Z \otimes \omega \to \omega$ there is a unique morphism $g: Z \to E$ in $_k\mathcal{M}$ such that the diagram



commutes.

(iv) Let E and $\rho: E \otimes \omega \to \omega$ be as in (iii). Show that E has a k-algebra structure such that ω becomes a functor into the category of left E-modules.