Reconstruction Theorems in Category Theory - Exercise Sheet 6

University of Siegen
Dr. Florian Heiderich

Fall Term 2016/2017
Due on January 26, 2017

Exercise 1.

Let k be a field and H be a k-Hopf algebra and H' be the k-bialgebra obtained from H with the same k-coalgebra structure but with respect to the k-algebra structure given by multiplication $\mathrm{m}' := \mathrm{m} \circ \tau$, where $\tau\colon H \otimes_k H \to H \otimes_k H$ is defined by $\tau(a \otimes b) = b \otimes a$, and the unit of H. Show that H' has an antipode S' if and only if the antipode S of H is bijective. In this case $S' = S^{-1}$.

Exercise 2.

Let k be a field and H be a Hopf algebra over k. Let ${}_{H/k}\mathfrak{m}$ be the monoidal category of left H-modules of finite dimension over k.

(i) Show that ${}_{H/k}\mathfrak{m}$ is left rigid.

(ii) Show that ${}_{H/k}\mathfrak{m}$ is right rigid if the antipode S of H is invertible (with respect to the composition).

Exercise 3.

Let k be a field, H be a k-Hopf algebra and ${}_{H/k}\mathcal{M}$ be the monoidal category of left H-modules with tensor product $\otimes_k$.

(i) Show that ${}_{H/k}\mathcal{M}$ is left closed.

(ii) Show that ${}_{H/k}\mathcal{M}$ is right closed if the antipode of H is invertible (with respect to the composition).

Exercise 4.

Let k be a field and ${}_k\mathcal{M}$ be the category of k-vector spaces.

(i) Let V be a k-vector space. Show that there exists a k-vector space E together with a morphism $\rho\colon E \otimes V \to V$ such that for every k-vector space Z and every morphism $\rho'\colon Z \otimes V \to V$ of k-vector spaces there exists a unique morphism $g\colon Z \to E$ of k-vector spaces such that the diagram

$$\begin{array}{ccc} Z \otimes V & & \\ \downarrow{\scriptstyle g \otimes \mathrm{id}_V} & \overset{\rho'}{\searrow} & \\ E \otimes V & \xrightarrow{\rho} & V \end{array}$$

commutes. Show that E and ρ are determined up to unique isomorphism.

(ii) Let E and $\rho\colon E \otimes V \to V$ be as in (i). Show that E has a k-algebra structure such that V becomes a left E-module with respect to ρ.

(iii) Let C be a small category and $\omega\colon \mathsf{C} \to {}_k\mathcal{M}$ be a functor. Show that there exists a k-vector space E and a natural transformation $\rho\colon E \otimes \omega \to \omega$ such that for any k-vector space Z and any natural transformation $\rho' \colon Z \otimes \omega \to \omega$ there is a unique morphism $g \colon Z \to E$ in ${}_k\mathcal{M}$ such that the diagram

$$\begin{array}{ccc} Z \otimes \omega & & \\ \downarrow{\scriptstyle g\otimes \mathrm{id}_\omega} & \searrow^{\rho'} & \\ E \otimes \omega & \xrightarrow{\rho} & \omega \end{array}$$

commutes.

(iv) Let E and $\rho \colon E \otimes \omega \to \omega$ be as in (iii). Show that E has a k-algebra structure such that ω becomes a functor into the category of left E-modules.