Reconstruction Theorems in Category Theory - Exercise Sheet 7

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Exercise 1.
Let k be a field.

(i) Show that if C is a k-coalgebra, then $\check{C} := {}_k\mathcal{M}(C, k)$ is a k-algebra.

(ii) Let $C := k[x] = \oplus_{i \in \mathbb{N}} kx^i$ be the free k-module with basis $\{1, x, x^2, ...\}$, considered as k-coalgebra with respect to the comultiplication Δ induced by

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

and counit ε induced by

$$\varepsilon(x^n) = \delta_{n,0}.$$

Determine the k-algebra $\check{C}$.

(iii) Show that if A is a finite dimensional k-algebra, then $\check{A} := {}_k\mathcal{M}(A, k)$ is a k-coalgebra.

(iv) Let FAlg_k be the category of finite-dimensional k-algebras and FCoAlg_k be the category of finite-dimensional k-coalgebras. Show that the functor from FCoAlg_k to FAlg_k that sends a finite-dimensional k-coalgebra (C, Δ, ε) to the k-algebra $\check{C} := {}_k\mathcal{M}(C, k)$ (see (i)) is an anti-equivalence of categories.

(v) Let C be a k-coalgebra. Show that a left (resp. right) C-comodule M is a left (resp. right) $\check{C}$-module. Show that the converse holds if C is finite dimensional over k.

(vi) Let C be a k-coalgebra.

(a) Show that $\mathcal{M}^C$ is a full subcategory of ${}_{\check{C}}\mathcal{M}$.

(b) Show that $\mathcal{M}^C$ is equivalent to ${}_{\check{C}}\mathcal{M}$ if C is of finite dimension over k.

(c) Provide an example of a k-coalgebra C such that $\mathcal{M}^C$ and ${}_{\check{C}}\mathcal{M}$ are not equivalent.

Exercise 2.
Let $(\mathsf{C}, \otimes, I, \Psi)$ be a rigid symmetric monoidal category. For an endomorphism f of an object X of C, we define the ***trace*** of f as the endomorphism of I that is given by

$$\mathrm{Tr}(f) := \varepsilon_X \circ \Psi_{X, \check{X}} \circ (f \otimes \mathrm{id}_{\check{X}}) \circ \eta_X.$$

(i) Show that $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$ for morphisms $f\colon X \to Y$ and $g\colon Y \to X$ in C.

(ii) Show that $\mathrm{Tr}(f_1 \otimes f_2) = \mathrm{Tr}(f_1) \circ \mathrm{Tr}(f_2) = \mathrm{Tr}(f_2) \circ \mathrm{Tr}(f_1)$ for morphisms $f_1\colon X_1 \to X_1$ and $f_2\colon X_2 \to X_2$ in C.