On Hasse-Schmidt rings and module algebras

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Abstract

Recently, Moosa and Scanlon introduced (iterative) Hasse-Schmidt systems $\mathcal{D}$ and, given such a Hasse-Schmidt system, they defined (iterative) $\mathcal{D}$-rings, generalizing rings with higher derivation as introduced by Hasse and Schmidt in 1937. We show that there is a bijection between Hasse-Schmidt systems $\mathcal{D}$ and cocommutative coalgebras D. For a given Hasse-Schmidt system $\mathcal{D}$ with associated coalgebra D we show that $\mathcal{D}$-rings are in bijection to algebras with a D-measuring on them. Under these correspondences iterative Hasse-Schmidt systems $\mathcal{D}$ correspond to cocommutative bialgebras D and iterative $\mathcal{D}$-rings correspond to D-module algebras.

Keywords: bialgebra, Hopf algebra, module algebra, derivation, endomorphism
2010 MSC: 12H05, 12H10, 13N15, 16T05

1. Introduction

In commutative algebra and related areas, especially differential algebra and difference algebra, rings are often equipped with extra structures such as derivations, higher derivations, endomorphisms or automorphisms. Similarities are apparent between the theory of rings with derivations and rings with endomorphisms, i.e. between differential rings and difference rings. There are at least two approaches that unify these structures. The first is that of D-measurings, where D is a coalgebra, which probably first appeared in the book [1] by Sweedler. Another approach was recently proposed by Moosa and Scanlon in [2] and [3].

In this note we show that to each Hasse-Schmidt system $\mathcal{D}$ in the sense of Moosa and Scanlon there is associated canonically a cocommutative coalgebra D. Then Hasse-Schmidt rings ($\mathcal{D}$-rings) are in bijection with algebras equipped with a D-measuring. If the Hasse-Schmidt system $\mathcal{D}$ is unital and iterative, the associated coalgebra D carries a natural structure of a bialgebra. In this case there is a 1-1 correspondence between unital iterative $\mathcal{D}$-rings and D-module algebras. Though we note that the definition of Hasse-Schmidt systems due to Moosa and Scanlon differs slightly from ours. In contrast to them, we do not require Hasse-Schmidt systems to be *unital*. This definition seems to be more natural due to the fact that a Hasse-Schmidt system $\mathcal{D}$ in our sense gives rise to a coalgebra D. If it is in addition unital, then D is equipped with a "unit map". Similarly an iterative structure on $\mathcal{D}$ yields

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[1] The author thanks La Fondation Sciences Mathématiques de Paris for financial support. He was also partially supported by the grant MTM2009-07024, Spanish Ministry of Education.

a "multiplication map" on D. If finally $\mathcal{D}$ is unital iterative, then these maps turn D into a bialgebra. Therefore the unital condition on $\mathcal{D}$ corresponds to a part of the algebra structure of D and is not strictly necessary. We note that a similar discrepancy pre-existed already in the definition of higher derivations: While a condition on the 0th higher derivation is made in [4], no such condition is present in [1].

The structure of this note is as follows: In section 2 we recall the definitions of measuring and of module algebras, while in section 3 we recall the basic definitions of (unital, iterative) Hasse-Schmidt systems $\mathcal{D}$ and given such a system, of (unital, iterative) $\mathcal{D}$-rings. In section 4 we construct a cocommutative coalgebra associated to every Hasse-Schmidt system. A construction inverse to this one is given in section 5. Given a Hasse-Schmidt system $\mathcal{D}$ with associated coalgebra D, we establish in section 6 a 1-1 correspondence between $\mathcal{D}$-rings and algebras with D-measuring. In the special case of a unital iterative Hasse-Schmidt system $\mathcal{D}$ with associated bialgebra D this correspondence restricts to one between unital iterative D-rings and D-module algebras. In section 7 it is explained that this correspondence gives rise to an isomorphism between the category of commutative $\mathcal{D}$-rings and the category of commutative k-algebras with D-measuring. In the case of iterative unital Hasse-Schmidt systems $\mathcal{D}$ with associated bialgebra D this isomorphism restricts to an isomorphism between the full subcategories of unital iterative commutative $\mathcal{D}$-rings and commutative D-module algebras. In section 8 we give some examples illustrating our results.

This article originates in section 2.3.9 of the author's doctoral thesis [5].

Notation: *We assume all rings and algebras to be unital and associative, but not necessarily to be commutative. Homomorphisms of algebras are assumed to respect the units. We further assume that all coalgebras are counital and coassociative, but not necessarily to be cocommutative. Homomorphisms of coalgebras are assumed to respect the counits. When we abbreviate an algebra* (A, m, η) *by* A*, then multiplication and unit will also be denoted by* m_A *and* η_A*, respectively. Similarly, when we abbreviate a coalgebra* (D, Δ, ε) *by* D*, then comultiplication and counit will also be denoted by* Δ_D *and* ε_D*, respectively. If* D *is a coalgebra and* $d \in D$*, then we use the* Σ*-notation* $\Delta_D(d) = \sum_{(d)} d_{(1)} \otimes d_{(2)}$ *(cf. [1, Section 1.2] or [6, 1.4.2]).*

We denote the category of algebras over a commutative ring R *by* Alg_R *and the category of left* R*-modules by* ${}_R\mathcal{M}$*.*

If C *is a category and* A *and* B *are objects in* C*, then we denote the class of morphisms from* A *to* B *in* C *by* $\mathsf{C}(A, B)$*. We denote the opposite category of* C *by* C^{op}*.*

The category of sets is denoted by Set*. If* A *and* B *are sets and* $a \in A$*, then we denote by* $\mathrm{ev}_a \colon \mathsf{Set}(A, B) \to B$ *the evaluation map, i.e.* $\mathrm{ev}_a(f) = f(a)$ *for all* $f \in \mathsf{Set}(A, B)$*. For elements* $a, b \in A$ *we denote by* $\delta_{a,b}$ *the Kronecker delta, i.e.* $\delta_{a,a} = 1$ *and* $\delta_{a,b} = 0$ *if* $a \neq b$*. We denote the tensor algebra of a module* M *by* $\mathrm{T}(M)$*.*

Let k *be a commutative ring.*

2. Module algebras

Given a k-coalgebra D, we recall the definition of D-measurings and of D-module algebras in the case where D is a k-bialgebra.

We first recall that for k-modules A, B and D there is an isomorphism of left k-modules

$$ {}_k\mathcal{M}(D \otimes_k A, B) \to {}_k\mathcal{M}(A, {}_k\mathcal{M}(D, B)), \quad \Psi \mapsto (a \mapsto (d \mapsto \Psi(d \otimes a))). \tag{2.1} $$

Lemma 2.1. *If $(D, \Delta_D, \varepsilon_D)$ is a k-coalgebra and $(B, \mathrm{m}_B, \eta_B)$ is a k-algebra, then the k-module ${}_k\mathcal{M}(D, B)$ becomes a k-algebra with respect to the* convolution product, *defined by*

$$f \cdot g \coloneqq \mathrm{m}_B \circ (f \otimes g) \circ \Delta_D$$

for $f, g \in {}_k\mathcal{M}(D, B)$, and unit element given by the composition

$$D \xrightarrow{\varepsilon_D} k \xrightarrow{\eta_B} B.$$

Furthermore, D is cocommutative if and only if ${}_k\mathcal{M}(D, B)$ is commutative for every commutative k-algebra B.

Proof. See for example [7, 1.3]. □

Proposition 2.2. *Let D be a k-coalgebra and let A and B be k-algebras. If Ψ is an element of ${}_k\mathcal{M}(D \otimes_k A, B)$ and $\rho \in {}_k\mathcal{M}(A, {}_k\mathcal{M}(D, B))$ is the image of Ψ under the isomorphism (2.1), then the following are equivalent:*

(1) ρ is a homomorphism of k-algebras,

(2) for all $d \in D$ and all $a, b \in A$

(a) $\Psi(d \otimes ab) = \sum_{(d)} \Psi(d_{(1)} \otimes a)\Psi(d_{(2)} \otimes b)$

(b) $\Psi(d \otimes 1_A) = \varepsilon_D(d) 1_B$ hold,

and

(3) the diagrams

$$\begin{array}{ccccc}
D \otimes_k A \otimes_k A & \xrightarrow{\mathrm{id}_D \otimes \mathrm{m}_A} & D \otimes_k A & \xrightarrow{\Psi} & B \\
\downarrow{\scriptstyle \Delta_D \otimes \mathrm{id}_A \otimes \mathrm{id}_A} & & & & \uparrow{\scriptstyle \mathrm{m}_B} \\
D \otimes_k D \otimes_k A \otimes_k A & \xrightarrow{\mathrm{id}_D \otimes \tau \otimes \mathrm{id}_A} & D \otimes_k A \otimes_k D \otimes_k A & \xrightarrow{\Psi \otimes \Psi} & B \otimes_k B,
\end{array}$$

where $\tau \colon D \otimes_k A \to A \otimes_k D$ is defined by $\tau(d \otimes a) = a \otimes d$ for all $a \in A$ and $d \in D$, and

$$\begin{array}{ccc}
D \otimes_k k & \xrightarrow{\varepsilon_D \otimes \eta_B} & k \otimes_k B \\
\downarrow{\scriptstyle \mathrm{id}_D \otimes \eta_A} & & \downarrow{\scriptstyle \sim} \\
D \otimes_k A & \xrightarrow{\Psi} & B
\end{array}$$

commute.

Proof. The equivalence between (2) and (3) is clear and the one between (1) and (2) can be seen by expanding the definition of ρ and of the condition that ρ be a homomorphism of k-algebras as is worked out in detail in [1, Proposition 7.0.1]. □

Definition 2.3. *Let D be a k-coalgebra and A and B be k-algebras. If $\Psi \in {}_k\mathcal{M}(D \otimes_k A, B)$, then we say that Ψ* measures *A to B if the equivalent conditions in proposition 2.2 are satisfied.*

If A_1, A_2, B_1 and B_2 are k-algebras, $\Psi_1 \colon D \otimes_k A_1 \to B_1$ measures A_1 to B_1 and $\Psi_2 \colon D \otimes_k A_2 \to B_2$ measures A_2 to B_2, then we say that homomorphisms $\varphi_A \colon A_1 \to A_2$ and $\varphi_B \colon B_1 \to B_2$ of k-algebras are compatible with the D-measurings *if the diagram*

$$\begin{array}{ccc} D \otimes_k A_1 & \xrightarrow{\Psi_1} & B_1 \\ \downarrow{\scriptstyle \mathrm{id}_D \otimes \varphi_A} & & \downarrow{\scriptstyle \varphi_B} \\ D \otimes_k A_2 & \xrightarrow{\Psi_2} & B_2 \end{array}$$

commutes.

The following lemmata are clear from the definitions.

Lemma 2.4. *Let D be a k-coalgebra and A_1, A_2, B_1 and B_2 be k-algebras. If $\Psi_1 \in {}_k\mathcal{M}(D \otimes_k A_1, B_1)$ measures A_1 to B_1 and $\Psi_2 \in {}_k\mathcal{M}(D \otimes_k A_2, B_2)$ measures A_2 to B_2 and ρ_1 and ρ_2 are the associated homomorphisms of k-algebras, then homomorphisms of k-algebras $\varphi_A \colon A_1 \to A_2$ and $\varphi_B \colon B_1 \to B_2$ are compatible with the D-measurings if and only if the diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\rho_1} & {}_k\mathcal{M}(D, B_1) \\ \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle {}_k\mathcal{M}(D, \varphi_B)} \\ A_2 & \xrightarrow{\rho_2} & {}_k\mathcal{M}(D, B_2) \end{array}$$

commutes.

Lemma 2.5. *Let D be a k-bialgebra and A be a k-algebra. If $\Psi \in {}_k\mathcal{M}(D \otimes_k A, A)$ and $\rho \colon A \to {}_k\mathcal{M}(D, A)$ is the homomorphism associated to Ψ via* (2.1)*, then Ψ makes A into a D-module if and only if the diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\rho} & {}_k\mathcal{M}(D, A) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle {}_k\mathcal{M}(D, \rho)} \\ {}_k\mathcal{M}(D, A) & \xrightarrow{{}_k\mathcal{M}(\mathrm{m}_D, A)} & {}_k\mathcal{M}(D \otimes_k D, A) \cong {}_k\mathcal{M}(D, {}_k\mathcal{M}(D, A)) \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\rho} & {}_k\mathcal{M}(D, A) \\ & \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle \mathrm{ev}_{1_D}} \\ & & A \end{array}$$

commute.

Definition 2.6. *Let A be a k-algebra, D be a k-bialgebra and let $\Psi \in {}_k\mathcal{M}(D \otimes_k A, A)$ measure A to itself. We say that Ψ is a* (left) D-module algebra structure *on A if Ψ makes A into a D-module (cf. lemma 2.5). The pair (A, Ψ) is then called* (left) D-module algebra.

A commutative D-module algebra *is a D-module algebra (A, Ψ) such that the k-algebra A is commutative.*

A homomorphism of D-module algebras *from (A_1, Ψ_1) to (A_2, Ψ_2) is a homomorphism of k-algebras $\varphi\colon A_1 \to A_2$ that fulfills the equivalent conditions of lemma 2.4 (with $B_1 = A_1$ and $B_2 = A_2$).*

Notation: *If $\Psi\colon D \otimes_k A \to B$ is a homomorphism of k-modules, then we denote by $\rho\colon A \to {}_k\mathcal{M}(D, B)$ the homomorphism corresponding to Ψ under the isomorphism* (2.1) *and vice versa. If $d \in D$ and $a \in A$, then we denote $\Psi(d \otimes a)$ also by $d.a$ or $d(a)$ if there is no danger of confusion.*

3. Moosa and Scanlon's $\mathcal{D}$-rings

In this section we recall the definition of (unital, iterative) Hasse-Schmidt systems and, given such a (unital, iterative) Hasse-Schmidt system $\mathcal{D}$, of (unital, iterative) $\mathcal{D}$-rings. These terms have been introduced by Moosa and Scanlon in [2] and [3]. Our exposition differs slightly from the original definition of Moosa and Scanlon in that we do not assume Hasse-Schmidt systems $\mathcal{D}$ and $\mathcal{D}$-rings to be unital in general.

Notation: *We denote by $\mathbb{S}$ the* standard ring scheme over k*, i.e. the k-scheme $\mathbb{S} \coloneqq \operatorname{Spec} k[x]$ regarded as a ring scheme by equipping for every commutative k-algebra A the set $\mathbb{S}(A) \cong A$ with the given ring structure of A.*

Definition 3.1 ([2, Definition 3.1] and [3, Definition 2.1])**.** *A* finite free commutative $\mathbb{S}$-algebra scheme *is an affine commutative $\mathbb{S}$-algebra scheme $\mathcal{E}$ that is isomorphic to $\mathbb{S}^l$ as $\mathbb{S}$-module scheme for some $l \in \mathbb{N}$.*

Remark 3.2. *We note that a choice of an isomorphism $\mathcal{E} \to \mathbb{S}^l$ is part of the definition of Moosa and Scanlon. This isomorphism is used in [2, Remark 3.2] to show that for every k-algebra R there is a canonical isomorphism $\mathcal{E}(R) \xrightarrow{\sim} R \otimes_k \mathcal{E}(k)$. However, this isomorphism does in fact not depend on the particular choice of an isomorphism $\mathcal{E} \to \mathbb{S}^l$.*

Definition 3.3 ([3, Definition 2.2])**.** *A* Hasse-Schmidt system *over k is a projective system of finite free commutative*[2] *$\mathbb{S}$-algebra schemes*

$$\mathcal{D} = (\pi_{m,n}\colon \mathcal{D}_m \to \mathcal{D}_n)_{m,n \in \mathbb{N}, n \leq m}$$

such that the $\pi_{m,n}$ are surjective morphisms of $\mathbb{S}$-algebra schemes. If in addition $\mathcal{D}_0 = \mathbb{S}$ holds, then we say that $\mathcal{D}$ is a unital *Hasse-Schmidt system.*

Definition 3.4 ([3, Definition 2.4])**.** *Let $\mathcal{D}$ be a Hasse-Schmidt system over k. Then a* commutative $\mathcal{D}$-ring *over k is a pair $(R, (E_n)_{n \in \mathbb{N}})$ consisting of a commutative k-algebra R and a family*

$$(E_n\colon R \to \mathcal{D}_n(R))_{n \in \mathbb{N}}$$

[2] It seems that Moosa and Scanlon assume implicitly that the algebra schemes occurring in the definition of Hasse-Schmidt systems are commutative, so we make this assumption here as well.

of k-algebra homomorphisms such that the diagram

$$
\begin{array}{ccc}
\mathcal{D}_m(R) & \xrightarrow{\pi_{m,n}(R)} & \mathcal{D}_n(R) \\
{\scriptstyle E_m}\nwarrow & & \nearrow{\scriptstyle E_n} \\
& R &
\end{array}
$$

commutes for all $n, m \in \mathbb{N}$ with $m \geq n$. If in addition $\mathcal{D}$ is a unital Hasse-Schmidt system, then we say that a commutative $\mathcal{D}$-ring $(R, (E_n)_{n\in\mathbb{N}})$ is unital *if $E_0 = \mathrm{id}$ holds.*

We recall that there is a finite free commutative $\mathbb{S}$-algebra scheme $\mathcal{D}_{(m,n)}$ defined by

$$\mathcal{D}_{(m,n)}(A) := \mathcal{D}_m(\mathcal{D}_n(A))$$

for all commutative k-algebras A (cf. [2, §4.2]).

Remark 3.5. *There is a slight difference in terminology in comparison with the work of Moosa and Scanlon. What is called a Hasse-Schmidt system (ring) in [3] is a* unital *Hasse-Schmidt system (ring) in our notation.*

Definition 3.6 ([3, Definition 2.17])**.** *An* iterative Hasse-Schmidt system *is a Hasse-Schmidt system $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ together with a family of closed immersions of $\mathbb{S}$-algebra schemes*[3]

$$\Delta = (\Delta_{(m,n)} \colon \mathcal{D}_{m+n} \to \mathcal{D}_{(m,n)})_{n,m\in\mathbb{N}} \tag{3.1}$$

such that the diagrams

$$
\begin{array}{ccc}
\mathcal{D}_{m+n} & \xrightarrow{\Delta_{(m,n)}} & \mathcal{D}_{(m,n)} \\
{\scriptstyle \pi_{m+n,m'+n'}}\downarrow & & \downarrow{\scriptstyle \pi_{(m,n),(m',n')}} \\
\mathcal{D}_{m'+n'} & \xrightarrow{\Delta_{(m',n')}} & \mathcal{D}_{(m',n')},
\end{array}
\tag{3.2}
$$

where the morphism $\pi_{(m,n),(m',n')}$ is defined on A-points as the composition

$$\mathcal{D}_{(m,n)}(A) = \mathcal{D}_m(\mathcal{D}_n(A)) \xrightarrow{\mathcal{D}_m(\pi_{n,n'}(A))} \mathcal{D}_m(\mathcal{D}_{n'}(A)) \xrightarrow{\pi_{m,m'}(\mathcal{D}_{n'}(A))} \mathcal{D}_{m'}(\mathcal{D}_{n'}(A)) = \mathcal{D}_{(m',n')}(A),$$

and

$$
\begin{array}{ccc}
\mathcal{D}_{n+m+l}(A) & \xrightarrow{\Delta_{(n+m,l)}(A)} & \mathcal{D}_{n+m}(\mathcal{D}_l(A)) \\
{\scriptstyle \Delta_{(n,m+l)}(A)}\downarrow & & \downarrow{\scriptstyle \Delta_{(n,m)}(\mathcal{D}_l(A))} \\
\mathcal{D}_n(\mathcal{D}_{m+l}(A)) & \xrightarrow{\mathcal{D}_n(\Delta_{(m,l)}(A))} & \mathcal{D}_n(\mathcal{D}_m(\mathcal{D}_l(A)))
\end{array}
\tag{3.3}
$$

[3] Moosa and Scanlon only require $\Delta_{(m,n)}$ to be morphisms of ring schemes. It seems however natural to assume that they are morphisms of $\mathbb{S}$-algebra schemes and we need this in the proof of proposition 4.1.

commute for all commutative k-algebras A and all $m, m', n, n', l \in \mathbb{N}$ with $m' \leq m$ and $n' \leq n$.

A unital iterative Hasse-Schmidt system *is a unital Hasse-Schmidt system that is iterative with respect to a family of morphisms* (3.1) *such that*

$$\Delta_{(m,0)} = \Delta_{(0,m)} = \mathrm{id} \tag{3.4}$$

holds for all $m \in \mathbb{N}$.

Definition 3.7 ([3, Definition 2.17])**.** *If $(\mathcal{D}, \Delta)$ is an iterative Hasse-Schmidt system, then a* commutative Δ-iterative $\mathcal{D}$-ring *is a commutative $\mathcal{D}$-ring $(R, (E_n)_{n\in\mathbb{N}})$ such that the diagram*

$$\begin{array}{ccc} \mathcal{D}_{m+n}(R) & \xrightarrow{\Delta_{(m,n)}(R)} & \mathcal{D}_m(\mathcal{D}_n(R)) \\ & {}_{E_{m+n}}\nwarrow \quad \nearrow_{E_{(m,n)}} & \\ & R & \end{array}$$

commutes for all $m, n \in \mathbb{N}$, where $E_{(m,n)} \colon R \to \mathcal{D}_m(\mathcal{D}_n(R))$ is defined as the composition

$$R \xrightarrow{E_m} \mathcal{D}_m(R) \xrightarrow{\mathcal{D}_m(E_n)} \mathcal{D}_m(\mathcal{D}_n(R)).$$

A commutative unital Δ-iterative $\mathcal{D}$-ring *is a commutative unital $\mathcal{D}$-ring $(R, (E_n)_{n\in\mathbb{N}})$ that is iterative.*

4. The coalgebra D associated to a Hasse-Schmidt system $\mathcal{D}$

In this section we will construct for every Hasse-Schmidt system $\mathcal{D}$ a coalgebra D. If $\mathcal{D}$ is unital or iterative, then D will have additional structures.

Proposition 4.1. *Let $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ be a Hasse-Schmidt system over k.*

(1) Then

$$D := \varinjlim_{n\in\mathbb{N}} \mathcal{D}_n(k)^*,$$

where we denote by $D_n := \mathcal{D}_n(k)^*$ the dual ${}_k\mathcal{M}(\mathcal{D}_n(k), k)$ of the k-module $\mathcal{D}_n(k)$, becomes naturally a cocommutative k-coalgebra. For every commutative k-algebra A there is an isomorphism of k-algebras

$${}_k\mathcal{M}(D, A) \xrightarrow{\sim} \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(A). \tag{4.1}$$

(2) If $\mathcal{D}$ is unital, then there is a canonically defined homomorphism of k-coalgebras

$$\eta \colon k \to D. \tag{4.2}$$

(3) If $\mathcal{D}$ is iterative with respect to

$$\Delta = (\Delta_{(m,n)} \colon \mathcal{D}_{m+n} \to \mathcal{D}_{(m,n)})_{m,n\in\mathbb{N}}, \tag{4.3}$$

then the morphisms (4.3) *induce a homomorphism of k-coalgebras*

$$\mathrm{m} \colon D \otimes_k D \to D$$

that defines an associative multiplication on D and fulfills

$$\mathrm{m}(D_n \otimes D_m) = D_{n+m}.$$

(4) *If $\mathcal{D}$ is a unital iterative Hasse-Schmidt system, then D becomes a k-bialgebra with unit η and multiplication* m.

Proof. We denote the transition maps of $\mathcal{D}$ by $\pi_{m,n}\colon \mathcal{D}_m \to \mathcal{D}_n$ for all $m, n \in \mathbb{N}$ with $m \geq n$. The structure of a commutative k-algebra on the finitely generated free k-module $\mathcal{D}_n(k)$ induces a structure of a cocommutative k-coalgebra on the dual $\mathcal{D}_n(k)^*$ for all $n \in \mathbb{N}$ and the homomorphisms of k-algebras $\pi_{m,n}(k)\colon \mathcal{D}_m(k) \to \mathcal{D}_n(k)$ induce homomorphisms of k-coalgebras $\pi_{m,n}(k)^*\colon \mathcal{D}_n(k)^* \to \mathcal{D}_m(k)^*$ forming a direct system in the category of k-coalgebras. The k-coalgebra structures on $\mathcal{D}_n(k)^*, n \in \mathbb{N}$ induce a k-coalgebra structure on $D \coloneqq \varinjlim_{n\in\mathbb{N}} \mathcal{D}_n(k)^*$, which again is cocommutative. For every commutative k-algebra A we have, recalling [2, Remark 3.2],

$$\begin{aligned}
{}_k\mathcal{M}(D, A) &= {}_k\mathcal{M}(\varinjlim_{n\in\mathbb{N}} \mathcal{D}_n(k)^*, A) \\
&\cong \varprojlim_{n\in\mathbb{N}} {}_k\mathcal{M}(\mathcal{D}_n(k)^*, A) \\
&\cong \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(k) \otimes_k A \\
&\cong \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(A).
\end{aligned}$$

If $\mathcal{D}$ is unital, then the homomorphisms of k-algebras

$$\pi_{n,0}(k)\colon \mathcal{D}_n(k) \to \mathcal{D}_0(k) = k$$

give rise to homomorphisms of k-coalgebras $k \to \mathcal{D}_n(k)^*$ and thus to $\eta\colon k \to \mathcal{D}_n(k)^* \to D$ (this composition does not depend on $n \in \mathbb{N}$). The compatibility of the comultiplication Δ with η follows from the compatibility of $\pi_{n,0}$ with the multiplication of $\mathcal{D}_n(k)$ and $\mathcal{D}_0(k)$. The compatibility of the counit ε with η follows from the compatibility of $\pi_{n,0}(k)$ with the units of $\mathcal{D}_n(k)$ and $\mathcal{D}_0(k)$.

We assume now that $\mathcal{D}$ is iterative with respect to (4.3). By [2, Remark 4.10], there is a canonical isomorphism of k-algebras

$$\mathcal{D}_{(m,n)}(k) \xrightarrow{\sim} \mathcal{D}_m(k) \otimes_k \mathcal{D}_n(k). \tag{4.4}$$

The homomorphisms of k-algebras

$$\mathcal{D}_{m+n}(k) \xrightarrow{\Delta_{(m,n)}(k)} \mathcal{D}_{(m,n)}(k) \xrightarrow{\quad\sim\quad} \mathcal{D}_m(k) \otimes_k \mathcal{D}_n(k)$$

induce homomorphisms of k-coalgebras

$$\mathcal{D}_m(k)^* \otimes_k \mathcal{D}_n(k)^* \xrightarrow{\quad\sim\quad} \mathcal{D}_{(m,n)}(k)^* \xrightarrow{\Delta_{(m,n)}(k)^*} \mathcal{D}_{m+n}(k)^*$$

for all $m, n \in \mathbb{N}$. These give rise to a homomorphism of k-coalgebras

$$\mathrm{m}\colon D \otimes_k D \to D,$$

which makes the diagram

$$\begin{array}{ccc}
D \otimes_k D & \xrightarrow{\qquad\mathrm{m}\qquad} & D \\
\uparrow & & \uparrow \\
\mathcal{D}_m(k)^* \otimes_k \mathcal{D}_n(k)^* & \xrightarrow{\Delta_{(m,n)}(k)^*} & \mathcal{D}_{m+n}(k)^*
\end{array}$$

commutative for all $m, n \in \mathbb{N}$. The property $\mathrm{m}(D_n \otimes D_m) = D_{n+m}$ follows from the fact that $\Delta_{(m,n)}$ is a closed immersion.

From the property (3.3) in the definition of iterative Hasse-Schmidt systems we see, using implicitly the isomorphisms (4.4), that the diagram

$$\begin{array}{ccc} \mathcal{D}_n(k)\otimes_k \mathcal{D}_m(k)\otimes_k \mathcal{D}_l(k) & \xleftarrow{\Delta_{n,m}(k)\otimes_k \mathcal{D}_l(k)} & \mathcal{D}_{n+m}(k)\otimes_k\mathcal{D}_l(k) \\ {\scriptstyle \mathcal{D}_n(k)\otimes_k\Delta_{m,l}(k)}\uparrow & & \uparrow{\scriptstyle\Delta_{n+m,l}(k)} \\ \mathcal{D}_n(k)\otimes_k\mathcal{D}_{m+l}(k) & \xleftarrow{\Delta_{n,m+l}(k)} & \mathcal{D}_{n+m+l}(k) \end{array}$$

commutes for all $n, m, l \in \mathbb{N}$ and thus dually the inner rectangle of

$$\begin{array}{ccccc} D\otimes_k D\otimes_k D & & \xrightarrow{\mathrm{m}\otimes D} & & D\otimes_k D \\ & \nwarrow & & \nearrow & \\ {\scriptstyle D\otimes\mathrm{m}}\downarrow & \mathcal{D}_n(k)^*\otimes_k \mathcal{D}_m(k)^*\otimes_k \mathcal{D}_l(k)^* & \xrightarrow{\Delta_{n,m}(k)^*\otimes\mathcal{D}_l(k)^*} & \mathcal{D}_{n+m}(k)^*\otimes_k\mathcal{D}_l(k)^* & \downarrow{\scriptstyle\mathrm{m}} \\ & {\scriptstyle \mathcal{D}_n(k)^*\otimes\Delta_{m,l}(k)^*}\downarrow & & \downarrow{\scriptstyle\Delta_{n+m,l}(k)^*} & \\ & \mathcal{D}_n(k)^*\otimes_k\mathcal{D}_{m+l}(k)^* & \xrightarrow{\Delta_{n,m+l}(k)^*} & \mathcal{D}_{n+m+l}(k)^* & \\ & \swarrow & & \searrow & \\ D\otimes_k D & & \xrightarrow{\mathrm{m}} & & D \end{array}$$

commutes too. Using the universal property of the direct limit

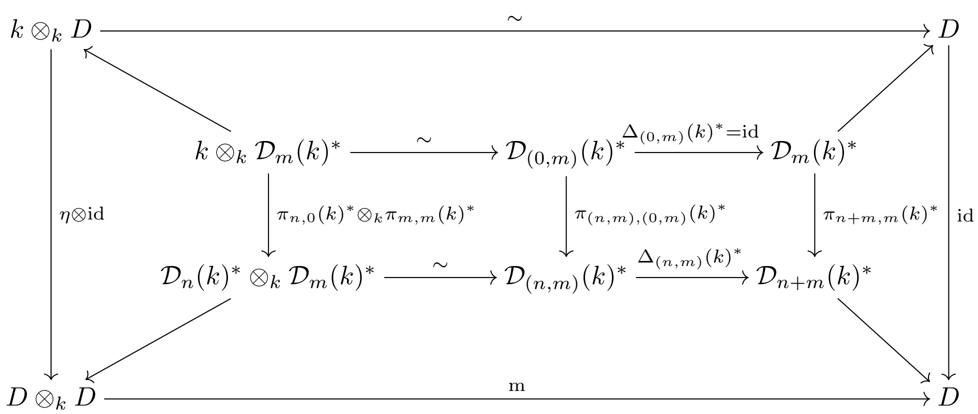
$$D\otimes_k D\otimes_k D \cong \varinjlim_{n,m,l\in\mathbb{N}} \mathcal{D}_n(k)^*\otimes_k \mathcal{D}_m(k)^*\otimes_k \mathcal{D}_l(k)^*$$

we conclude that the outer rectangle also commutes, i.e. that m is associative. The compatibility of the comultiplication Δ with m follows from the compatibility of $\Delta_{(m,n)}$ with the multiplication of $\mathcal{D}_{m+n}$ and $\mathcal{D}_{(m,n)}$. The compatibility of the counit ε with m follows from the compatibility of $\Delta_{(m,n)}$ with the units of $\mathcal{D}_{m+n}$ and $\mathcal{D}_{(m,n)}$.

Finally, we assume that $\mathcal{D}$ is a unital iterative Hasse-Schmidt system. By the properties (3.2) and (3.4) the diagram

$$\begin{array}{ccccc} k\otimes_k\mathcal{D}_m(k) & \xleftarrow{\sim} & \mathcal{D}_{(0,m)}(k) & \xleftarrow{\Delta_{(0,m)}(k)=\mathrm{id}} & \mathcal{D}_m(k) \\ {\scriptstyle \pi_{n,0}(k)\otimes_k\pi_{m,m}(k)}\uparrow & & {\scriptstyle \pi_{(n,m),(0,m)}(k)}\uparrow & & \uparrow{\scriptstyle\pi_{n+m,m}(k)} \\ \mathcal{D}_n(k)\otimes_k\mathcal{D}_m(k) & \xleftarrow{\sim} & \mathcal{D}_{(n,m)}(k) & \xleftarrow{\Delta_{(n,m)}(k)} & \mathcal{D}_{n+m}(k). \end{array}$$

commutes for all $n, m \in \mathbb{N}$. Therefore, dually the inner rectangles of



commute and, again by the universal property of the direct limit, the outer rectangle commutes too. This means that η is a left unit for the multiplication m. Similarly, one can show that η is a right unit. □

5. The Hasse-Schmidt system $\mathcal{D}$ associated to a cocommutative coalgebra D

In this section we conversely associate a Hasse-Schmidt system $\mathcal{D}$ to a cocommutative k-coalgebra D. If D is moreover a k-bialgebra, then $\mathcal{D}$ will become a unital iterative Hasse-Schmidt system.

Proposition 5.1. *Let D be a cocommutative k-coalgebra that is the direct limit of finite free k-subcoalgebras $(D_n)_{n\in\mathbb{N}}$ with $D_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$.*

(1) Then a Hasse-Schmidt system $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ is defined by finite free commutative $\mathbb{S}$-algebra schemes $\mathcal{D}_n$ that are defined by

$$\mathcal{D}_n(A) := {}_k\mathcal{M}(D_n, A)$$

for all commutative k-algebras A.

(2) If additionally $D_0 = k$, then $\mathcal{D}$ is unital.

(3) If there is an associative composition law on D given by a homomorphism $\mathrm{m}\colon D \otimes_k D \to D$ of k-coalgebras and if $\mathrm{m}(D_n \otimes_k D_m) = D_{n+m}$, then $\mathcal{D}$ is iterative with respect to the family of morphisms

$$\Delta = (\Delta_{(n,m)}\colon D_{n+m} \to \mathcal{D}_{(n,m)})_{n,m\in\mathbb{N}}$$

that are defined as the compositions

$$\mathcal{D}_{n+m}(A) = {}_k\mathcal{M}(D_{n+m}, A) \xrightarrow{{}_k\mathcal{M}(\mathrm{m},A)} {}_k\mathcal{M}(D_n \otimes_k D_m, A) \xrightarrow{\sim} {}_k\mathcal{M}(D_n, {}_k\mathcal{M}(D_m, A)) = \mathcal{D}_n(\mathcal{D}_m(A))$$

for all commutative k-algebras A.

(4) If D is a k-bialgebra with respect to the unit $\eta\colon k \xrightarrow{\mathrm{id}} D_0 \subseteq D$ and multiplication m, then $\mathcal{D}$ is a unital iterative Hasse-Schmidt system.

Proof. The functors $\mathcal{D}_n$ are finite free commutative $\mathbb{S}$-algebra schemes, since the D_n are cocommutative k-coalgebras that are finitely generated free as k-modules and since

$$\mathcal{D}_n(A) = {}_k\mathcal{M}(D_n, A) \cong \mathsf{Alg}_k(\mathrm{T}(D_n), A).$$

Since $D_n \subseteq D_m$ for all $n \leq m$, we obtain compatible surjections

$$\pi_{m,n}(A)\colon \mathcal{D}_m(A) \to \mathcal{D}_n(A).$$

Therefore $\mathcal{D} \coloneqq (\mathcal{D}_n)_{n\in\mathbb{N}}$ is a Hasse-Schmidt system over k.

If $D_0 = k$, then we obtain $\mathcal{D}_0 = \mathbb{S}$.

If $\mathrm{m}\colon D \otimes_k D \to D$ is a homomorphism of k-coalgebras defining an associative composition law on D such that $\mathrm{m}(D_n \otimes_k D_m) = D_{n+m}$, then morphisms

$$\Delta = (\Delta_{(n,m)}\colon \mathcal{D}_{n+m} \to \mathcal{D}_{(n,m)})_{n,m\in\mathbb{N}}$$

are defined on A-points by the homomorphisms of A-algebras

$$\Delta_{(n,m)}(A)\colon \mathcal{D}_{n+m}(A) = {}_k\mathcal{M}(D_{n+m}, A) \to {}_k\mathcal{M}(D_n \otimes_k D_m, A) = {}_k\mathcal{M}(D_n, {}_k\mathcal{M}(D_m, A)) = \mathcal{D}_n(\mathcal{D}_m(A))$$

that are induced by the restriction of the multiplication $\mathrm{m}\colon D_n \otimes_k D_m \to D_{n+m}$. We notice that

$$\mathcal{D}_{n+m}(A) = {}_k\mathcal{M}(D_{n+m}, A) \cong \mathsf{Alg}_k(\mathrm{T}(D_{n+m}), A)$$

and

$$\mathcal{D}_{(n,m)}(A) = \mathcal{D}_n(\mathcal{D}_m(A)) = {}_k\mathcal{M}(D_n, {}_k\mathcal{M}(D_m, A)) = {}_k\mathcal{M}(D_n \otimes_k D_m, A) \cong \mathsf{Alg}_k(\mathrm{T}(D_n \otimes_k D_m), A)$$

Therefore the morphisms $\Delta_{(n,m)}$ are induced by the homomorphisms

$$\mathrm{T}(\mathrm{m})\colon\ \mathrm{T}(D_n \otimes_k D_m) \to \mathrm{T}(D_{n+m})$$

on the coordinate rings, which are surjective, since $\mathrm{m}\colon D_n \otimes_k D_m \to D_{n+m}$ is surjective. Therefore the morphisms $\Delta_{(n,m)}$ are closed immersions. The conditions (3.2) and (3.3) are fulfilled since the multiplication is compatible with the restriction and since it is associative.

If finally D is a k-bialgebra with $D_0 = \eta(k)$, then m restricts to the canonical isomorphisms $D_m \otimes_k D_0 \xrightarrow{\sim} D_m \otimes_k k \xrightarrow{\sim} D_m$ and $D_0 \otimes_k D_m \xrightarrow{\sim} k \otimes_k D_m \xrightarrow{\sim} D_m$, since η is a left and right unit for m, and therefore (3.4) holds. □

Remark 5.2. *The procedures in propositions 4.1 and 5.1 are inverse to each other.*

6. $\mathcal{D}$-rings and D-measurings

Remark 6.1. *Let $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ be a Hasse-Schmidt system over k. Then due to the universal property of the inverse limit there is a bijection between the set of commutative $\mathcal{D}$-rings and the set of pairs (R, E) where R is a commutative k-algebra and $E\colon R \to \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R)$ is a k-algebra homomorphism. If the Hasse-Schmidt system $\mathcal{D}$ is unital, then the homomorphism $E\colon R \to \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R)$ renders the composition*

$$R \xrightarrow{E} \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) \to \mathcal{D}_0(R) = R$$

into the identity if and only if the corresponding commutative $\mathcal{D}$-ring is unital. By abuse of notation we denote a commutative $\mathcal{D}$-ring $(R, (E_n)_{n\in\mathbb{N}})$ also by (R, E).

Proposition 6.2. *Let $\mathcal{D}$ be a Hasse-Schmidt system over k and $D = \varinjlim_{n\in\mathbb{N}} \mathcal{D}_n(k)^*$ the associated k-coalgebra (see proposition 4.1).*

(1) If (R, E) is a commutative $\mathcal{D}$-ring over k, then to E there is associated canonically a D-measuring $\rho\colon R \to {}_k\mathcal{M}(D, R)$ from R to itself and the diagram

$$\begin{array}{ccc} & & \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) \\ & \overset{E}{\nearrow} & \uparrow \\ R & & \\ & \underset{\rho}{\searrow} & \\ & & {}_k\mathcal{M}(D, R) \end{array}$$

commutes, where the vertical arrow is the isomorphism (4.1).

If $\mathcal{D}$ is unital and (R, E) is a unital commutative $\mathcal{D}$-ring, then the composition

$$R \xrightarrow{\rho} {}_k\mathcal{M}(D, R) \xrightarrow{\mathrm{ev}_{1_D}} R \tag{6.1}$$

is the identity on R, where 1_D denotes the image of $1 \in k$ under the homomorphism η (cf. (4.2)).

If $\mathcal{D}$ is an iterative Hasse-Schmidt system and (R, E) is an iterative commutative $\mathcal{D}$-ring, then the D-measuring $\rho\colon R \to {}_k\mathcal{M}(D, R)$ renders the diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad\quad\rho\quad\quad} & {}_k\mathcal{M}(D, R) \\ \big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle {}_k\mathcal{M}(D,\rho)} \\ {}_k\mathcal{M}(D, R) & \xrightarrow{{}_k\mathcal{M}(\mathrm{m},R)} & {}_k\mathcal{M}(D \otimes_k D, R) \cong {}_k\mathcal{M}(D, {}_k\mathcal{M}(D, R)) \end{array} \tag{6.2}$$

commutative, where m is the homomorphism constructed in proposition 4.1 (3).

If finally $\mathcal{D}$ is a unital iterative Hasse-Schmidt system and (R, E) a commutative unital iterative $\mathcal{D}$-ring, then ρ is a D-module algebra structure, where D is the k-bialgebra constructed in proposition 4.1 (4).

(2) Conversely, to every D-measuring $\rho\colon R \to {}_k\mathcal{M}(D, R)$ from a commutative k-algebra R to itself there is canonically associated a $\mathcal{D}$-ring (R, E).

If $D_0 = k$ and $\eta\colon k \to D$ is the inclusion of $D_0 = k$ into D and the composition (6.1) is the identity, where $1_D \coloneqq \eta(1)$, then the $\mathcal{D}$-ring (R, E) is unital.

If there is a homomorphism of k-coalgebras $\mathrm{m}\colon D \otimes_k D \to D$ that defines an associative composition law on D and if ρ makes the diagram (6.2) commutative, then the $\mathcal{D}$-ring (R, E) is iterative.

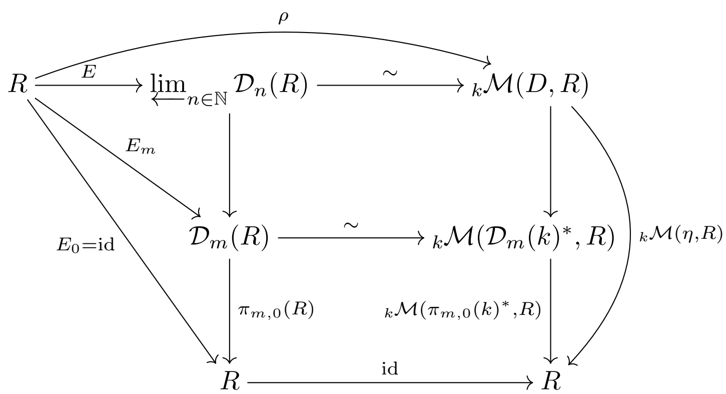
If D is a k-bialgebra and ρ is a D-module algebra structure on R, then the $\mathcal{D}$-ring (R, E) is unital iterative.

The constructions in (1) and (2) are inverse to each other.

Proof. Given a commutative $\mathcal{D}$-ring (R, E), we define a D-measuring $\rho\colon R \to {}_k\mathcal{M}(D, R)$ from R to itself as the composition of the homomorphisms of k-algebras

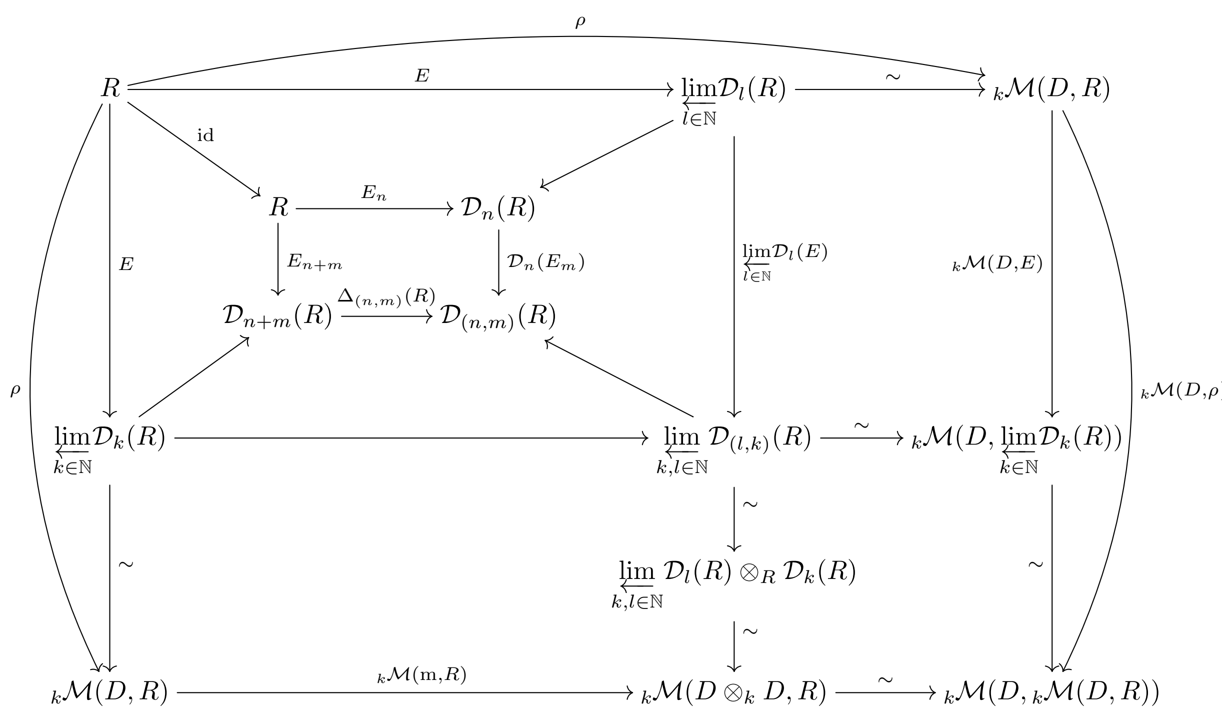
$$R \xrightarrow{E} \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) \xrightarrow{(4.1)} {}_k\mathcal{M}(D, R).$$

If $\mathcal{D}$ is a unital Hasse-Schmidt system and (R, E) is a unital commutative $\mathcal{D}$-ring, then by the definition of the unit of D, for every $m \in \mathbb{N}$ the diagram



commutes.

If $(\mathcal{D}, \Delta)$ is an iterative Hasse-Schmidt system and (R, E) is a commutative Δ-iterative $\mathcal{D}$-ring, then by the definition of m on D (cf. proposition 4.1(3)) the diagram



commutes for all $m, n \in \mathbb{N}$.

If $(\mathcal{D}, \Delta)$ is a unital iterative Hasse-Schmidt system and (R, E) is a commutative unital Δ-iterative $\mathcal{D}$-ring, then from the commutativity of the two previous diagrams we conclude that $\rho\colon R \to {}_k\mathcal{M}(D, R)$ is a D-module algebra structure on R.

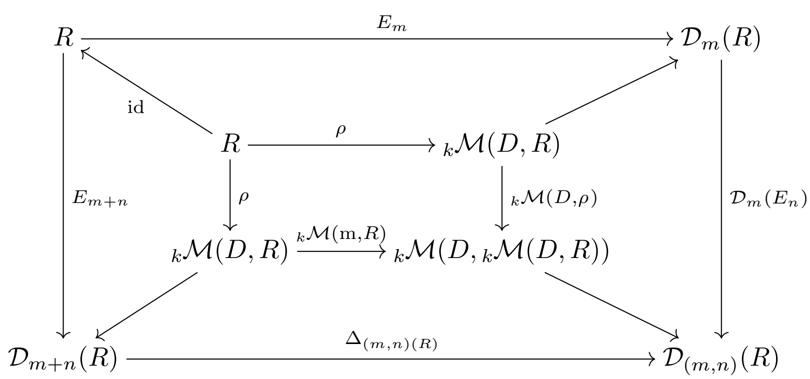
If, conversely, $\rho\colon R \to {}_k\mathcal{M}(D, R)$ is a D-measuring from R to itself, then for every $m \in \mathbb{N}$ we define a homomorphism of k-algebras $E_m\colon R \to \mathcal{D}_m(R)$ as the composition

$$R \xrightarrow{\rho} {}_k\mathcal{M}(D, R) \xrightarrow{\sim} \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) \to \mathcal{D}_m(R).$$

Then by definition the maps $(E_n)_{n\in\mathbb{N}}$ fulfill the relations $E_n = \pi_{m,n}(R) \circ E_m$ for all $m \geq n$. Consequently, the family $(E_n)_{n\in\mathbb{N}}$ defines a $\mathcal{D}$-ring structure on R.

If additionally $D_0 = k$ and $1_D \coloneqq \eta(1)$ renders (6.1) into the identity, then $\mathcal{D}_0(R) = R$ and E_0 is the identity, i.e. (R, E) is a unital commutative $\mathcal{D}$-ring.

If there is a homomorphism of k-coalgebras $\mathrm{m}\colon D \otimes_k D \to D$ that defines an associative composition law on D and ρ makes the diagram (6.2) commutative, then the inner rectangle of the diagram



commutes, and thus also the outer for all $m, n \in \mathbb{N}$. This means that (R, E) is a commutative Δ-iterative $\mathcal{D}$-ring.

If moreover D is a k-bialgebra and ρ is a D-module algebra structure on R, by the previous (R, E) is a commutative unital Δ-iterative $\mathcal{D}$-ring.

Using the identification described in remark 6.1, we see that the passage between the (unital, iterative) $\mathcal{D}$-ring structure E on R and (unital, associative) D-measurings ρ on R is given by composition with the isomorphism $\varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) \xrightarrow{\sim} {}_k\mathcal{M}(D, R)$ and its inverse. Therefore, the constructions in (1) and (2) are inverse to each other. □

7. Isomorphism of categories

In [3] the authors do not define morphisms between commutative $\mathcal{D}$-rings over k. Though, if $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ is a Hasse-Schmidt system over k and if $(R, (E_n)_{n\in\mathbb{N}})$ and $(S, (F_n)_{n\in\mathbb{N}})$ are commutative $\mathcal{D}$-rings, then a morphism from $(R, (E_n)_{n\in\mathbb{N}})$ to $(S, (F_n)_{n\in\mathbb{N}})$ can be defined as a homomorphism of k-algebras $\varphi\colon R \to S$ such that $\mathcal{D}_n(\varphi) \circ E_n = F_n \circ \varphi$ holds for all $n \in \mathbb{N}$. Then a homomorphism of k-algebras $\varphi\colon R \to S$ is a morphism of $\mathcal{D}$-rings if and only if the induced morphism $\varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(\varphi)\colon \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) \to \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(S)$ fulfills $F \circ \varphi = \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(\varphi) \circ E$.

If D is the k-coalgebra associated to $\mathcal{D}$ by proposition 4.1, then the diagram

$$\begin{array}{ccc}
\varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(R) & \xrightarrow{\sim} & {}_k\mathcal{M}(D,R) \\
\Big\downarrow {\scriptstyle \varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(\varphi)} & & \Big\downarrow {\scriptstyle {}_k\mathcal{M}(D,\varphi)} \\
\varprojlim_{n\in\mathbb{N}} \mathcal{D}_n(S) & \xrightarrow{\sim} & {}_k\mathcal{M}(D,S),
\end{array}$$

commutes, where the horizontal arrows are the isomorphisms from proposition 4.1. So we see that there is a bijection between the homomorphisms of $\mathcal{D}$-rings from $(R,(E_n)_{n\in\mathbb{N}})$ to $(S,(F_n)_{n\in\mathbb{N}})$ and the homomorphisms of k-algebras with D-measuring from R to S, where both algebras are equipped with the D-measuring induced by their $\mathcal{D}$-ring structures. Together with proposition 6.2 we see that the category of commutative $\mathcal{D}$-rings and the category of commutative algebras with D-measuring are isomorphic. If $(\mathcal{D},\Delta)$ is a unital iterative Hasse-Schmidt system and D is the associated k-bialgebra, then this isomorphism induces an isomorphism between the category of commutative unital Δ-iterative $\mathcal{D}$-rings and the category of commutative D-module algebras.

8. Examples

In this section we illustrate our results in the cases of Hasse-Schmidt derivations and endomorphisms.

Example 8.1 (Hasse-Schmidt derivations). *Hasse-Schmidt derivations are described as follows: Let $D := k\langle \theta^{(l)} \mid l \in \mathbb{N}\rangle$ be the free k-module with basis $(\theta^{(l)})_{l\in\mathbb{N}}$. It becomes a k-algebra with multiplication defined by $\theta^{(m)}\theta^{(l)} := \binom{m+l}{k}\theta^{(m+l)}$ and unit $1 := \theta^{(0)}$ and a cocommutative k-coalgebra via*

$$\Delta(\theta^{(l)}) := \sum_{l_1+l_2=l} \theta^{(l_1)} \otimes \theta^{(l_2)} \qquad \text{and} \qquad \varepsilon(\theta^{(l)}) := \delta_{l,0}.$$

A family of finite free k-subcoalgebras is defined by $D_n := k\langle\theta^{(0)},\dots,\theta^{(n)}\rangle$ for all $n \in \mathbb{N}$. We note that ${}_k\mathcal{M}(D_n, A) \cong A[\![t]\!]/(t^{n+1})$ and ${}_k\mathcal{M}(D, A) \cong A[\![t]\!]$ for every commutative k-algebra A. The associated Hasse-Schmidt system $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ is given by the finite free $\mathbb{S}$-algebra schemes $\mathcal{D}_n$ defined by

$$\mathcal{D}_n(A) = {}_k\mathcal{M}(D_n, A) = \mathsf{Alg}_k(\mathrm{T}(D_n), A) \cong A[\![t]\!]/(t^{n+1})$$

for commutative k-algebras A. We note that $\mathrm{T}(D_n) \cong k[\theta^{(0)},\dots,\theta^{(n)}]$. The Hasse-Schmidt system $\mathcal{D}$ is unital and iterative with respect to the family $\Delta = (\Delta_{(n,m)})_{n,m\in\mathbb{N}}$ of morphisms

$$\Delta_{(n,m)} \colon \mathcal{D}_{n+m} \to \mathcal{D}_{(n,m)}$$

defined by

$$\mathcal{D}_{n+m}(A) = A[\![t]\!]/(t^{n+m+1}) \to (A[\![u]\!]/(u^m+1))[\![v]\!]/(v^{n+1}) = \mathcal{D}_n(\mathcal{D}_m(A)), \quad t \mapsto u+v.$$

The corresponding homomorphism on the coordinate rings is given as

$$\mathrm{T}(D_n) \otimes \mathrm{T}(D_m) \to \mathrm{T}(D_{n+m}), \quad \theta^{(l_1)} \otimes \theta^{(l_2)} \mapsto \binom{l_1+l_2}{l_1}\theta^{(l_1+l_2)}$$

for all $l_1 \in \{0, \dots, n\}$ and $l_2 \in \{0, \dots, m\}$.

In this situation commutative $\mathcal{D}$-rings are commutative k-algebras R equipped with a family of k-linear homomorphisms $\theta^{(l)}\colon R \to R$ such that

$$\theta^{(l)}(1) = \delta_{l,0} \quad \text{and} \quad \theta^{(l)}(ab) = \sum_{l_1+l_2=l} \theta^{(l_1)}(a)\theta^{(l_2)}(b)$$

for all $a, b \in R$, i.e. commutative k-algebras equipped with a higher derivation. By proposition 6.2 they can be equivalently described as commutative k-algebras together with a D-measuring on them or by a homomorphism of commutative k-algebras

$$\theta\colon R \to R[\![t]\!].$$

A commutative $\mathcal{D}$-ring is unital if and only if the composition

$$R \xrightarrow{\theta} R[\![t]\!] \xrightarrow{t\mapsto 0} R \tag{8.1}$$

of the associated homomorphism $\theta\colon R \to R[\![t]\!]$ with the homomorphism of R-algebras $R[\![t]\!] \to R$, that maps t to 0, is the identity on R.

A commutative $\mathcal{D}$-rings is Δ-iterative if and only if the associated homomorphisms $\theta\colon R \to R[\![t]\!]$ makes the following diagram commutative

$$\begin{array}{ccc} R & \xrightarrow{\theta} & R[\![u]\!] \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta[\![u]\!]} \\ R[\![t]\!] & \xrightarrow{t\mapsto u+v} & R[\![u,v]\!]. \end{array} \tag{8.2}$$

Consequently, commutative unital Δ-iterative $\mathcal{D}$-rings are in 1-1 correspondence with commutative k-algebras equipped with an iterative derivation, i.e. a homomorphism $\theta\colon R \to R[\![t]\!]$ that renders the composition (8.1) *into the identity and makes the diagram* (8.2) *commutative.*

Remark 8.2. *A higher derivation as defined by Sweedler in [1] corresponds to a D-measuring on a commutative ring (or a commutative $\mathcal{D}$-ring) as in the previous example. The definition of higher derivations by Matsumura in [4] corresponds to* unital *$\mathcal{D}$-rings in the last example.*

Example 8.3 (Endomorphisms)**.** *Endomorphisms on rings can be described as follows: Let*

$$D := k\langle \sigma_i \mid i \in \mathbb{N} \rangle$$

be the free k-module generated by $(\sigma_i)_{i\in\mathbb{N}}$ with k-algebra structure defined by $\sigma_i\sigma_j := \sigma_{i+j}$ and $1 := \sigma_0$ and cocommutative k-coalgebra structure defined by $\Delta(\sigma_i) = \sigma_i \otimes \sigma_i$ and $\varepsilon(\sigma_i) = 1$ for all $i, j \in \mathbb{N}$. We define k-subcoalgebras D_n of D as the free k-modules generated by $\sigma_0, \sigma_1, \dots, \sigma_n$. The corresponding Hasse-Schmidt system $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ is given by

$$\mathcal{D}_n(A) = {}_k\mathcal{M}(D_n, A) \cong A^{n+1}$$

for all commutative k-algebras A, where A^{n+1} is a k-algebra with pointwise addition and multiplication. The homomorphism $\pi_{m,n}(A)\colon \mathcal{D}_m(A) \to \mathcal{D}_n(A)$ is given by the projection to the first $n+1$ factors.

The Hasse-Schmidt system $\mathcal{D}$ becomes iterative with respect to the morphisms

$$\Delta_{(n,m)} \colon \mathcal{D}_{n+m}(A) \to \mathcal{D}_n(\mathcal{D}_m(A))$$

that are induced by the homomorphisms of k-algebras

$$\mathrm{T}(D_n) \otimes \mathrm{T}(D_m) \to \mathrm{T}(D_{n+m}), \quad \sigma_i \otimes \sigma_j \mapsto \sigma_{i+j},$$

which are surjective (and therefore $\Delta_{(n,m)}$ are closed immersions).

Commutative k-algebras R with D-measuring are in 1-1 correspondence with commutative k-algebras R that are equipped with a family of endomorphisms $(\sigma_i)_{i\in\mathbb{N}}$ of R. The corresponding commutative $\mathcal{D}$-ring (R, E) is given by a family of homomorphisms

$$E_n \colon R \to R^{n+1}$$

of k-algebras, defined by $E_n(r) = (\sigma_i(r))_{i=0,\dots,n}$ or equivalently by

$$E \colon R \to R^{\mathbb{N}}, \quad r \mapsto (\sigma_i(r))_{i\in\mathbb{N}}.$$

Using this notation, commutative D-module algebras (or equivalently: commutative unital Δ-iterative $\mathcal{D}$-rings) are in 1-1 correspondence with commutative k-algebras equipped with an endomorphism σ of k-algebras, such that $\sigma_i = \sigma^i$ for all $i \in \mathbb{N}$.

Other interesting structures on rings that can be described using D-measurings (resp. D-module algebras), where D is a cocommutative coalgebra (resp. bialgebra) include derivations, but also a modified version of the iterated q-difference operators introduced by Hardouin (cf. [8]) as explained by Masuoka (cf. [9]).

Remark 8.4. *It seems that while by using higher or iterative derivations certain problems in differential algebra in positive characteristic can be overcome, possibly similar problems in difference algebra can be resolved by the consideration of systems of endomorphisms or of higher powers of an endomorphisms. Indications for this appeared also in [10] and [11]. In the view of this note, in both cases this is achieved by considering certain "iterative" structures.*

Acknowledgments

I would like to thank Michael Wibmer for calling my attention to a little inaccuracy in a preliminary version of this article as well as the referee for helpful comments.

References

[1] M. E. Sweedler, Hopf algebras, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

[2] R. Moosa, T. Scanlon, Jet and prolongation spaces, J. Inst. Math. Jussieu 9 (2) (2010) 391–430. `doi:10.1017/S1474748010000010`.
URL `http://dx.doi.org/10.1017/S1474748010000010`

[3] R. Moosa, T. Scanlon, Generalized Hasse-Schmidt varieties and their jet spaces, Proc. Lond. Math. Soc. (3) 103 (2) (2011) 197–234. doi:10.1112/plms/pdq055.
URL http://dx.doi.org/10.1112/plms/pdq055

[4] H. Matsumura, Commutative Ring Theory, Cambridge University Press, Cambridge-New York, 1989.

[5] F. Heiderich, Galois theory of module fields, Ph.D. thesis, Universitat de Barcelona, Departament d'Àlgebra i Geometria, available from: http://hdl.handle.net/10803/674 (2010).

[6] S. Montgomery, Hopf algebras and their actions on rings, Vol. 82 of CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1993.

[7] T. Brzezinski, R. Wisbauer, Corings and comodules, Vol. 309 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003.

[8] C. Hardouin, Iterative q-difference Galois theory, J. Reine Angew. Math. 644 (2010) 101–144. doi:10.1515/CRELLE.2010.053.
URL http://dx.doi.org/10.1515/CRELLE.2010.053

[9] A. Masuoka, The $\times_r$-bialgebra associated with an iterative q-difference ring, preprint (2010).

[10] Z. Chatzidakis, E. Hrushovski, Y. Peterzil, Model theory of difference fields. II. Periodic ideals and the trichotomy in all characteristics, Proc. London Math. Soc. (3) 85 (2) (2002) 257–311. doi:10.1112/S0024611502013576.
URL http://dx.doi.org/10.1112/S0024611502013576

[11] M. Wibmer, A Chevalley theorem for difference equations, Math. Ann. 354 (4) (2012) 1369–1396. doi:10.1007/s00208-011-0770-0.
URL http://dx.doi.org/10.1007/s00208-011-0770-0