# GENERALIZED DIFFERENTIALS AND PROLONGATION SPACES

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Abstract. Given a coalgebra $D$, a commutative algebra $A$, a commutative $A$-algebra $B$ and a measuring $\psi\colon D\otimes A \to B$, we define an algebra $\Omega^D_{B/(A,\psi)}$ that generalizes the symmetric algebra over the module of Kähler differentials $\mathrm{Sym}_B(\Omega^1_{B/A})$. We show that the spectrum of $\Omega^D_{B/(A,\psi)}$ is isomorphic to a prolongation space as defined by Moosa and Scanlon, providing a direct construction of the latter. These prolongation spaces generalize those of Gillet, Rosen and Vojta. The universal prolongations of differential and difference kernels can also be recovered from our generalized differentials. When $D$ is moreover a bialgebra, our generalized differentials provide a unified approach to the prolongations of commutative rings, unifying the well-known constructions in the differential and difference case.

## Introduction

The literature contains several definitions of prolongations spaces. Most of them generalize the tangent bundle. Recently, a rather general definition of prolongation spaces was introduced by Moosa and Scanlon (cf. [MS10], [MS11]). While these prolongation spaces are defined in terms of Weil restrictions, we provide a direct construction in the case of affine schemes and show that they generalize the prolongation spaces defined previously by Buium, Rosen and Vojta (cf. [Bui93], [Ros08] and [Voj07]). As the tangent bundle can be defined in terms of the Kähler differentials, one can construct the prolongation spaces of Buium, Vojta and Rosen using algebras of (higher) differentials. We propose a definition of generalized differentials that unifies Kähler differentials as well as the divided differentials defined by Vojta and the higher differentials as defined by Rosen. The spectra of our generalized differentials realize the prolongation spaces due to Moosa and Scanlon in the case of affine schemes.

Let $f\colon A \to B$ be a commutative $A$-algebra. Recall that the Kähler differentials $\Omega^1_{B/A}$ of $B$ over $A$ together with $\mathrm{d}\colon B \to \Omega^1_{B/A}$ have the universal property that for every $B$-module $M$ and every $A$-derivation $\partial\colon B \to M$ there exists a unique morphism of $B$-modules $\phi\colon \Omega^1_{B/A} \to M$ such that $\partial = \phi \circ \mathrm{d}$, giving rise to a bijection

$$\mathrm{Der}_A(B, M) \cong {}_B\mathcal{M}(\Omega^1_{B/A}, M),$$

where ${}_B\mathcal{M}(\Omega^1_{B/A}, M)$ denotes the homomorphisms of $B$-modules from $\Omega^1_{B/A}$ to $M$. If $g\colon B \to R$ is a commutative $B$-algebra, then this bijection induces a bijection

$$\mathrm{Der}_A(B, R) \cong \mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/A}), R), \tag{0.1}$$

where $\mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/A}), R)$ denotes the homomorphisms of $B$-algebras from $\mathrm{Sym}_B(\Omega^1_{B/A})$ to $R$. If $\delta_A\colon A \to A$ is a derivation on $A$, the derivation $\mathrm{d}\colon B \to \mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})$ induces a derivation $\mathrm{d}\colon B \to \mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{\delta_A}$, where $I_{\delta_A}$ is the ideal of $\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})$ generated by $\mathrm{d} f(a) - f(\delta_A(a))$ for all $a \in A$, and the bijection (0.1) restricts to a bijection

$$\{\partial \in \mathrm{Der}_A(B, R) \mid \partial \circ f = g \circ f \circ \delta_A\} \cong \mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{\delta_A}, R). \tag{0.2}$$

In positive characteristic, higher derivations are often more suitable than classical derivations. Vojta and Rosen introduce differentials for higher derivations and generalize the bijections (0.1) and (0.2), respectively: Vojta defines for every $m \in \mathbb{N}$ a $B$-algebra $\mathrm{HS}^m_{B/A}$ such that for every commutative $A$-algebra $R$ there is a bijection

$$\mathsf{Alg}_A(\mathrm{HS}^m_{B/A}, R) \cong \mathsf{Alg}_A(B, R[t]/(t^{m+1})), \tag{0.3}$$

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the set at the right hand side describing higher derivations of length $m$ from $B$ to $R$ over $A$, cf. [Voj07]. Given a higher differential ring $(A, \delta_A = (\delta_A^{(i)})_{i\in\mathbb{N}})$ and a commutative $A$-algebra $B$, Rosen defines a $B$-algebra $\mathrm{HS}^m_{B/(A,\delta_A)}$ so that there is a bijection

$$\mathsf{Alg}_A(\mathrm{HS}^m_{B/(A,\delta_A)}, R) \cong \mathsf{Alg}_A(B, R[t]/(t^{m+1})), \tag{0.4}$$

where the $A$-algebra structure on $R[t]/(t^{m+1})$ is induced by the homomorphism $\theta\colon A \to A[t]/(t^{m+1})$, $a \mapsto \sum_{i=0}^m \delta_A^{(i)}(a)t^i$, cf. [Ros08]. Rosen's approach is the most general one of the aforementioned, specializing to the others as indicated by the arrows in the diagram

$$\begin{tikzcd}
\mathrm{HS}^m_{B/A}\ \text{(Vojta)} \arrow[d, "m=1", "\text{Remark 1.8}"'] & & \mathrm{HS}^m_{B/(A,\delta_A)}\ \text{(Rosen)} \arrow[ll, "\delta_A\ \text{trivial}"', "\text{Remark 1.16 (1)}"] \arrow[d, "m=1\ \text{and}\ \delta_A^{(0)} = \mathrm{id}_A", "\text{Remark 1.16 (2)}"'] \\
\mathrm{Sym}_B(\Omega^1_{B/A}) \ = \ \mathrm{HS}^1_{B/A} & & \mathrm{HS}^1_{B/(A,\delta_A)} = \mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{\delta_A^{(1)}}. \arrow[ll, "\delta_A^{(1)}\ \text{trivial}"']
\end{tikzcd}$$

These objects share the property of being universal objects for certain classes of (higher) derivations.

Note that a derivation on a commutative ring $A$ can be equivalently defined as a homomorphism of rings $\rho\colon A \to A[t]/(t^2)$ such that $\mathrm{ev}_0 \circ \rho = \mathrm{id}_A$, where $\mathrm{ev}_0\colon A[t]/(t^2) \to A$ is the homomorphism of $A$-algebras defined by $\mathrm{ev}_0(t) = 0$. Similarly, (unital) higher derivations $(\delta_A^{(i)})_{i=0,\dots,m}$ on $A$ can be defined as a homomorphism of rings $\rho\colon A \to A[t]/(t^{m+1})$ such that $\mathrm{ev}_0 \circ \rho = \mathrm{id}_A$. The definition of prolongation spaces Moosa and Scanlon is in terms of commutative $\mathcal{E}$-rings, where $\mathcal{E}$ is a finite free $\mathbb{S}$-algebra scheme over a commutative ring $k$. A commutative $\mathcal{E}$-ring is a commutative ring $A$ together with a homomorphism of $k$-algebras $e\colon A \to \mathcal{E}(A)$. This generalizes higher derivations, $\mathcal{E}(A)$ replacing $A[t]/(t^{m+1})$. In this article we mainly use $D$-measurings, where $D$ is a coalgebras, instead of $\mathcal{E}$-rings. This is equivalent to their framework as shown in [Hei13b]: Given a finite free $\mathbb{S}$-algebra scheme $\mathcal{E}$ over $k$, we define $D := \mathcal{E}(k)^*$. The $\mathbb{S}$-algebra structure on $\mathcal{E}$ induces a $k$-coalgebra structure on $D$ and commutative $\mathcal{E}$-rings $e\colon A \to \mathcal{E}(A)$ are equivalent to $D$-measurings $\psi\colon D \otimes_k A \to A$ from $A$ to itself. The $k$-algebra $\mathcal{E}(A)$ is isomorphic to the set ${}_k\mathcal{M}(D, A)$ of homomorphisms of $k$-modules from $D$ to $A$, which becomes a $k$-algebra thanks to the $k$-coalgebra structure on $D$. A $D$-measuring $\psi\colon D \otimes_k A \to A$ is equivalent to a homomorphism of $k$-algebras $\rho\colon A \to {}_k\mathcal{M}(D, A)$ and the isomorphism $\mathcal{E}(A) \cong {}_k\mathcal{M}(D, A)$ allows to pass from $\mathcal{E}$-ring structures on $A$ to $D$-measurings and vice versa.

More generally, given a commutative $k$-algebra $A$, a $D$-measuring $\psi_A\colon D \otimes_k A \to A$, and two commutative $A$-algebras $B$ and $R$, we can consider homomorphisms of $A$-algebras $P\colon B \to {}_k\mathcal{M}(D, R)$, a notion generalizing higher derivations form $B$ to $R$ over $A$. For our generalized differentials $\Omega^D_{B/(A,\psi_A)}$ there is a homomorphism of $A$-algebras $\rho_u\colon B \to {}_k\mathcal{M}(D, \Omega^D_{B/A})$ such that for every homomorphism of $A$-algebras $B \to {}_k\mathcal{M}(D, R)$ there is a unique homomorphism of $A$-algebras $\phi\colon \Omega^D_{B/(A,\psi_A)} \to R$ such that $P = \phi \circ \rho_u$, giving rise to a bijection

$$\mathsf{Alg}_A(B, {}_k\mathcal{M}(D, R)) \cong \mathsf{Alg}_A(\Omega^D_{B/(A,\psi_A)}, R), \quad P \mapsto \phi$$

that generalizes (0.3) and (0.4), cf. proposition 3.2.

From our generalized differentials we also recover universal prolongations of differential kernels due to Johnson (cf. [Joh82], [Joh85]) and difference kernels (cf. [Wib12], [Wib13]).

The forgetful functor from the category of commutative differential algebras over a given commutative differential ring $(A, \delta_A)$ to the category of commutative $A$-algebras has a left adjoint, as shown by Gillet, cf. [Gil02]. A similar result for commutative unital iterative differential algebras is due to Rosen: The forgetful functor from the category of commutative unital iterative differential algebras over a given commutative unital iterative differential ring $(A, \delta_A)$ to the category of commutative $(A, \delta_A)$-algebras (i.e. the category commutative algebras $f\colon A \to B$ such that $\operatorname{Ker} f$ is an $\delta_A$-ideal) has a left adjoint. Similarly, the forgetful functor from the category of commutative difference algebras over a given commutative difference field $(A, \sigma_A)$ to the category of commutative $A$-algebras has a left adjoint, as shown by Wibmer, cf. [Wib13]. If the above mentioned $k$-coalgebra $D$ is a $k$-bialgebra and $A$ is a commutative $D$-module algebra, we use our generalized differentials again to unify and generalize these results (cf. proposition 3.9).

The simplest kind of a prolongation space is the tangent bundle. In the case of an affine scheme $X = \operatorname{Spec} B$ over $Y = \operatorname{Spec} A$, the tangent bundle of $X$ over $Y$ is $\mathrm{T}_{X/Y} \coloneqq \operatorname{Spec} \operatorname{Sym}_B(\Omega^1_{B/A})$. Vojta defines, given a morphism of schemes $X \to Y$, the scheme of $m$-jet differentials $J_m(X/Y)$ of $X$ over $Y$, such that for every $Y$-scheme $Z$ there is an isomorphism

$$\mathsf{Sch}_Y(Z \times_{\mathbb{Z}} \mathbb{Z}[t]/(t^{m+1}), X) \cong \mathsf{Sch}_Y(Z, J_m(X/Y)).$$

Buium defines, given a derivation $\delta_A \colon A \to A$ on a commutative ring $A$ and scheme $X$ over $Y = \operatorname{Spec} A$, an $A$-scheme $jet_m(X/Y, \delta_A)$ such that for every $A$-scheme $Z$ there is a bijection

$$\mathsf{Sch}_A(Z \times_A A[t]/(t^{m+1}), X) \xrightarrow{\sim} \mathsf{Sch}_A(Z, jet_m(X/Y, \delta_A)),$$

where the $A$-scheme structures of $Z \times_A A[t]/(t^{m+1})$ is induced by the exponential map $e \colon A \to A[t]/(t^{m+1}), a \mapsto \sum_{i=0}^{m} \frac{\delta_A^i(a)}{i!} t^i$. Buium's definition specializes to the one of Vojta if the derivation $\delta_A$ is trivial. Rosen generalizes Buium's definition by using higher derivations. Given a finite free commutative $\mathbb{S}$-algebra scheme $\mathcal{E}$ over $k$, a commutative $\mathcal{E}$-ring $e \colon A \to \mathcal{E}(A)$ and a scheme $X$ over $Y = \operatorname{Spec} A$, Moosa and Scanlon define the prolongation space $\tau(X, \mathcal{E}, e)$ of $X$ with respect to $e \colon A \to \mathcal{E}(A)$ as the Weil restriction of $X \times_A \mathcal{E}^e(A)$ from $\mathcal{E}(A)$ to $A$. Therefore for every $A$-scheme $Z$ there is a bijection

$$\mathsf{Sch}_A(Z \times_A \mathcal{E}(A), X) \cong \mathsf{Sch}_A(Z, \tau(X, \mathcal{E}, e)).$$

By taking $\mathcal{E}(A) \coloneqq A[t]/(t^{m+1})$ and $e \colon A \to \mathcal{E}(A)$ to be the ring homomorphism induced by a (higher) derivation, one recovers the definitions of the Buium's jet spaces and Rosen's prolongation spaces as well as the tangent bundle, Vojta's jet spaces if the (higher) derivation is trivial.

Moosa and Scanlon define their prolongation spaces as certain Weil restrictions. Our generalized differentials provide an alternative and more direct construction of their prolongation spaces, at least in the case of affine schemes. Although it is not explicitly mentioned in their articles, it seems that they assume the finite free $\mathbb{S}$-algebra schemes $\mathcal{E}$ to be commutative, which is equivalent to the coalgebra $D$ associated to $\mathcal{E}$ to be cocommutative. Here we try not to impose this condition when it is not necessary, since operators like skew-derivations are described as a $D$-measurings for coalgebras $D$ that are not cocommutative. We note that commutative rings with iterative $q$-difference operators, as introduced by Hardouin in [Har10], can be described as $D$-module algebras for a cocommutative bialgebra $D$, as Masuoka and Yanagawa show, cf. [MY13].

Our interest in generalized differentials arose from the use of $D$-measurings and $D$-module algebras in Galois theories of functional equations, cf. [Tak89], [AM05], [AMT09], [Hei10], [Hei], [Hei13a].

This article is organized as follows: In the first section we begin with a review of Kähler differentials and a version of Kähler differentials relative to a derivation on the base ring. We recall higher derivations, introduced by Hasse and Schmidt (cf. [HS37]), which generalize classical derivations. Then we review divided differentials as defined by Vojta [Voj07] and higher differentials as defined by Rosen [Ros08]. We also recall differential prolongations due to Gillet [Gil02] and difference prolongations due to Hrushovski, Tomasic and Wibmer, cf. [Hru04], [Tom11] and [Wib13]. Finally we also include the definition of differential kernels of Johnson [Joh82], [Joh85]) and of difference kernels due to Wibmer [Wib12].

In section 2 we briefly recall the notion of a measuring and of module algebras, which we use to define our generalized differentials.

In section 3 we define generalized differentials. We prove universal properties of these generalized differentials and show how they generalize and unify several of the objects introduced in the first section. We also show functorial properties of our generalized differentials.

Section 4 recalls several definitions of prolongation spaces. The most well-known of them is the tangent bundle. We recall the definitions of Vojta's scheme of jet differentials, Buium's jet spaces and Rosen's prolongation spaces and see how they generalize the tangent bundle. We also observe how these jet and prolongation spaces can be constructed using some of the objects introduced in the first section.

Finally, in section 5 we recall the definition of prolongation spaces due to Moosa and Scanlon ([MS10], [MS11]) and provide a new construction of them in terms of our generalized differentials. We show how they specialize to the spaces introduced in section 4.

**Notation:** *We assume all rings and algebras to be unital and associative, but not necessarily to be commutative. Homomorphisms of algebras are assumed to respect the units and modules over (unital) rings are assumed to be* *unitary. We further assume that all coalgebras are counital and coassociative, but not necessarily to be cocommutative. Homomorphisms of coalgebras are assumed to respect the counits. If* $(D, \Delta, \varepsilon)$ *is a coalgebra and* $d \in D$*, then we use the Sweedler notation and denote* $\Delta(d)$ *by* $\sum_{(d)} d_{(1)} \otimes d_{(2)}$*. Let* $R$ *be a commutative ring. We denote the category of algebras over* $R$ *by* $\mathsf{Alg}_R$ *and the category of left* $R$*-modules by* ${}_R\mathcal{M}$*. The category of commutative* $R$*-algebras is denoted by* $\mathsf{CAlg}_R$*. We denote the symmetric algebra of an* $R$*-module* $M$ *by* $\mathrm{Sym}_R(M)$*. The category of schemes over* $R$ *is denoted by* $\mathsf{Sch}_R$*. An* $R$-ring *is a ring together with a ring homomorphism from* $R$ *into it.*

*If* $\mathsf{C}$ *is a category and* $A$ *and* $B$ *are objects in* $\mathsf{C}$*, then we denote the class of morphisms from* $A$ *to* $B$ *in* $\mathsf{C}$ *by* $\mathsf{C}(A, B)$*.*

*The category of sets is denoted by* $\mathsf{Set}$*. If* $A$ *and* $B$ *are sets and* $a \in A$*, then we denote by* $\mathrm{ev}_a \colon \mathsf{Set}(A, B) \to B$ *the evaluation map, i.e.* $\mathrm{ev}_a(f) = f(a)$ *for all* $f \in \mathsf{Set}(A, B)$*. For elements* $a, b \in A$ *we denote by* $\delta_{a,b}$ *the Kronecker delta, i.e.* $\delta_{a,a} = 1$ *and* $\delta_{a,b} = 0$ *if* $a \neq b$*.*

*We denote by* $\mathbb{N}$ *the set of natural numbers (including* $0$*) and by* $\mathbb{Z}$ *the integers. Let* $k$ *be a commutative ring.*

## 1. Review of differentials, differential- and difference kernels and of prolongations

This section is of introductory nature and we do not claim originality of most of its definitions and results. We first recall the classical Kähler differentials and a version of them relative to a derivation on the base ring. The latter is probably known, but have not found any reference. Then we recall the definition of higher derivations. We show how Vojta's divided differentials and a similar object, introduced by Rosen, generalize the Kähler differentials and the above mentioned relative version of them. We recall two results on differential and difference prolongations of commutative rings. Finally we state the definitions of differential kernels due to Johnson and of difference kernels due to Wibmer.

1.1. **Kähler differentials.** Given a commutative ring $A$ and a commutative $A$-algebra $f \colon A \to B$, a *module of Kähler differentials of* $B$ *over* $A$ is a $B$-module $\Omega^1_{B/A}$ together with an $A$-derivation $\mathrm{d} \colon B \to \Omega^1_{B/A}$, which satisfies the following universal property: For every $B$-module $M$ and every $A$-derivation $\partial \colon B \to M$ there exists a unique morphism of $B$-modules $\phi \colon \Omega^1_{B/A} \to M$ such that $\partial = \phi \circ \mathrm{d}$. Therefore there is an isomorphism of $B$-modules

$$\mathrm{Der}_A(B, M) \xrightarrow{\sim} {}_B\mathcal{M}(\Omega^1_{B/A}, M), \tag{1.1}$$

where $\mathrm{Der}_A(B, M)$ denotes the $A$-derivations from $B$ to $M$. By the universal property the module of Kähler differentials is unique. It also exists and there are two well-known constructions.

First, the Kähler differentials can be constructed by taking $\Omega^1_{B/A}$ to be the quotient of the free $B$-module generated by $\{\mathrm{d}\, b \mid b \in B\}$ by the submodule generated by the elements

$$\mathrm{d}(b + b') - \mathrm{d}\, b - \mathrm{d}\, b', \quad \mathrm{d}(bb') - b\, \mathrm{d}\, b' - b'\, \mathrm{d}\, b \quad \text{and} \quad \mathrm{d}\, f(a)$$

for all $b, b' \in B$ and $a \in A$. The $A$-derivation $\mathrm{d} \colon B \to \Omega^1_{B/A}$ is defined by sending $b$ to the image of $\mathrm{d}\, b$ in $\Omega^1_{B/A}$, which we denote by abuse of notation again by $\mathrm{d}\, b$, for all $b \in B$. The universal property is fulfilled by construction.

A second construction is as follows: If $\mathrm{m} \colon B \otimes_A B \to B$ is the multiplication of the $A$-algebra $B$ and $I = \mathrm{Ker}\, \mathrm{m}$ is its kernel, then $\mathrm{d} \colon B \to I/I^2, b \mapsto [b \otimes 1 - 1 \otimes b]$ is a module of Kähler differentials, cf. [Mat80, p. 182] for instance.

Later we use the isomorphism (1.1) in the case where $M$ is a commutative $B$-algebra, in the form

$$\mathrm{Der}_A(B, M) \xrightarrow{\sim} \mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/A}), M), \tag{1.2}$$

where $\mathrm{Sym}_B(\Omega^1_{B/A})$ denotes the symmetric algebra of $\Omega^1_{B/A}$ over $B$.

**Remark 1.1.** *There is an isomorphism of* $B$*-algebras*

$$\mathrm{Sym}_B(\Omega^1_{B/A}) \cong B[\mathrm{d}\, b \mid b \in B]/I,$$

*where* $I$ *is the ideal generated by*

$$\mathrm{d}(b + b') - \mathrm{d}\, b - \mathrm{d}\, b', \quad \mathrm{d}(bb') - b\, \mathrm{d}\, b' - b'\, \mathrm{d}\, b \quad \textit{and} \quad \mathrm{d}\, f(a)$$

*for all* $b, b' \in B$ *and* $a \in A$*.*

### 1.2. **Kähler differentials relative to a derivation.**

**Notation:** *Let $A$ be a commutative ring, $f\colon A \to B$ be a commutative $A$-algebra and $\delta_A\colon A \to A$ be a derivation.*

If $g\colon B \to R$ is a commutative $B$-algebra, then we denote the set of derivations $\partial\colon B \to R$ extending $\delta_A$ by

$$\mathrm{Der}_{\delta_A}(B,R) := \{\partial \in \mathrm{Der}_{\mathbb{Z}}(B,R) \mid \partial \circ f = g \circ f \circ \delta_A\}$$

The derivation $\mathrm{d}\colon B \to \Omega^1_{B/\mathbb{Z}}$ induces a derivation

$$\mathrm{d}\colon B \to \mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A},$$

where $I_{f\circ\delta_A}$ is the ideal of $\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})$ that is generated by $\mathrm{d}\,f(a) - f(\delta_A(a))$ for all $a \in A$. This derivation extends the given derivation $\delta_A\colon A \to A$ and fulfills the following universal property:

**Lemma 1.2.** *If $g\colon B \to R$ is a commutative $B$-algebra and $\partial \in \mathrm{Der}_{\delta_A}(B,R)$ is a derivation extending $\delta_A$, then there exists a unique homomorphism of $B$-algebras $\phi\colon \mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A} \to R$ such that $\phi \circ \mathrm{d} = \partial$, i.e. there is a bijection*

$$\mathrm{Der}_{\delta_A}(B,R) \cong \mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A}, R). \tag{1.3}$$

*Proof.* The homomorphism of $B$-algebras $\phi$ is uniquely defined by $\phi(\mathrm{d}\,b) := \partial(b)$. This is well defined, since $\phi(\mathrm{d}\,f(a) - f(\delta_A(a))) = \partial(f(a)) - g(f(\delta_A(a))) = 0$ for all $a \in A$. ☐

**Remark 1.3.** *We have*

$$\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A} \cong B[\mathrm{d}\,b \mid b \in B]/I,$$

*where $I$ is the ideal generated by*

$$\mathrm{d}(b+b') - \mathrm{d}\,b - \mathrm{d}\,b', \quad \mathrm{d}(bb') - b\,\mathrm{d}\,b' - b'\,\mathrm{d}\,b \quad \textit{and} \quad \mathrm{d}(f(a)) - f(\delta_A(a))$$

### 1.3. **Higher derivations.** We briefly recall the definition of higher derivations.

**Notation:** *Let $A$ be a commutative ring, $B$ and $R$ be commutative $A$-algebras and $m$ be a natural number or $\infty$.*

**Definition 1.4.** *A* higher derivation of length $m$ from $B$ to $R$ over $A$ *is a sequence*[1] $\delta = (\delta^{(0)}, \dots, \delta^{(m)})$ *of homomorphisms of $A$-modules $\delta^{(i)}\colon B \to R$ such that*

$$\delta^{(i)}(b_1 b_2) = \sum_{i_1+i_2=i} \delta^{(i_1)}(b_1)\cdot\delta^{(i_2)}(b_2) \qquad \textit{and} \qquad \delta^{(i)}(1) = \delta_{i,0}$$

*for all $b_1, b_2 \in B$ and all $i \in \{0,\dots,m\}$. We denote the set of higher derivations of length $m$ from $B$ to $R$ over $A$ by $\mathrm{Der}^m_A(B,R)$. A higher derivation $\delta = (\delta^{(0)}, \dots, \delta^{(m)})$ from $B$ to itself is called* unital *if $\delta^{(0)} = \mathrm{id}_B$ and it is called* iterative *if $\delta^{(i)} \circ \delta^{(j)} = \binom{i+j}{i}\delta^{(i+j)}$ for all $i, j \in \mathbb{N}$.*

**Remark 1.5.** *There is a discrepancy in the definition of higher derivations in the literature. While a condition on the 0th higher derivation is imposed in* [Mat89]*, no such condition is present in* [Swe69].

**Remark 1.6.** *We define $R_m$ to be the quotient $R[t]/(t^{m+1})$ of the polynomial ring $R[t]$ if $m \in \mathbb{N}$ and $R_\infty := R[\![t]\!]$. A sequence $\delta = (\delta^{(i)})_{i=0,\dots,m}$ of maps $\delta^{(i)}\colon B \to R$ is a higher derivation of length $m$ from $B$ to $R$ over $A$ if and only if the map*

$$\theta\colon B \to R_m, \quad b \mapsto \sum_{i=0}^{m} \delta^{(i)}(b)t^i$$

*is a homomorphism of $A$-algebras, where $R_m$ is considered as $A$-algebra via the composition of its natural $R$-algebra structure and the given $A$-algebra structure of $R$. Therefore there is a bijection*

$$\mathrm{Der}^m_A(B,R) \cong \mathsf{Alg}_A(B, R_m). \tag{1.4}$$

*Note that $\mathrm{Der}^1_A(B,R)$ does not coincide with $\mathrm{Der}_A(B,R)$ as defined in subsection 1.1. Their relation is explained in remark 1.10.*

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[1] If $m = \infty$, then we write by abuse of notation $\delta = (\delta^{(0)}, \dots, \delta^{(m)})$ instead of $\delta = (\delta^{(i)})_{i\in\mathbb{N}}$ and similarly in other situations.

### 1.4. Vojta's divided differentials.

**Notation:** *Let $A$ be a commutative ring and $f\colon A \to B$ be a commutative $A$-algebra.*

**Definition 1.7** ([Voj07, Definition 1.3])**.** *For every natural number $m$, the $B$-algebra of* divided differentials $\mathrm{HS}^m_{B/A}$ *is defined as the quotient*[2]

$$\mathrm{HS}^m_{B/A} \coloneqq B[b^{(i)} \mid b \in B, i \in \{0, \dots, m\}]/I,$$

*where $I$ is the ideal generated by the elements*

$$(b+b')^{(i)} - (b)^{(i)} - (b')^{(i)}, \quad (bb')^{(i)} - \sum_{i_1+i_2=i} b^{(i_1)} b'^{(i_2)}, \quad f(a)^{(j)} \quad \text{and} \quad (b)^{(0)} - b$$

*for all $b, b' \in B$, $a \in A$, $i \in \mathbb{N}$ and $j \geq 1$.*

*There is a higher derivation $\mathrm{d} = (\mathrm{d}^{(i)})_{i=0,\dots,m}$ of length $m$ from $B$ to $\mathrm{HS}^m_{B/A}$ over $A$ given by the $A$-linear maps*

$$\mathrm{d}^{(i)}\colon B \to \mathrm{HS}^m_{B/A}, \quad b \mapsto b^{(i)},$$

*which we call the* universal higher derivation *of length $m$.*

**Remark 1.8.** *We have isomorphisms of $B$-algebras $\mathrm{HS}^0_{B/A} \cong B$ and*

$$\mathrm{HS}^1_{B/A} \cong \mathrm{Sym}_B(\Omega^1_{B/A}), \tag{1.5}$$

*where $\Omega^1_{B/A}$ is the module of Kähler differentials of $B$ over $A$ as defined in subsection 1.1.*

The $B$-algebras $(\mathrm{HS}^m_{B/A})_{m \in \mathbb{N}}$ form a direct system and we define

$$\mathrm{HS}^\infty_{B/A} \coloneqq \varinjlim_{m \in \mathbb{N}} \mathrm{HS}^m_{B/A}\,.$$

**Proposition 1.9** ([Voj07, Corollary 1.8])**.** *For every commutative $A$-algebra $R$ there is an isomorphism*

$$\mathsf{Alg}_A(B, R[t]/(t^{m+1})) \cong \mathsf{Alg}_A(\mathrm{HS}^m_{B/A}, R). \tag{1.6}$$

Therefore the higher derivation $\mathrm{d} = (\mathrm{d}^{(i)}\colon B \to \mathrm{HS}^m_{B/A})_{i=0,\dots,m}$ is universal among all higher derivations of length $m$ over $A$ from $B$ to a commutative $A$-algebra $R$.

**Remark 1.10.** *Let $g\colon B \to R$ be a commutative $B$-algebra. Using the isomorphism* (1.5) *and considering $R[t]/(t^2)$ as $R$-algebra via the canonical $R$-algebra structure and as $B$-algebra via the composition $B \xrightarrow{g} R \to R[t]/(t^2)$, we obtain horizontal isomorphisms*

$$\begin{tikzcd}
\mathrm{Der}^1_A(B,R) \arrow[r, "(1.4)"] & \mathsf{Alg}_A(B, R[t]/(t^2)) \arrow[r, "(1.6)"] & \mathsf{Alg}_A(\mathrm{HS}^1_{B/A}, R) \arrow[r, "(1.5)"] & \mathsf{Alg}_A(\mathrm{Sym}_B(\Omega^1_{B/A}), R) \\
\mathrm{Der}_A(B,R) \arrow[u, hook] \arrow[r, "\sim"] \arrow[rrr, bend right, "(1.2)"'] & \{\theta \in \mathsf{Alg}_A(B, R[t]/(t^2)) \mid \varepsilon \circ \theta = g\} \arrow[u, hook] \arrow[r, "\sim"] & \mathsf{Alg}_B(\mathrm{HS}^1_{B/A}, R) \arrow[u, hook] \arrow[r, "(1.5)"] & \mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/A}), R), \arrow[u, hook]
\end{tikzcd}$$

*where $\varepsilon\colon R[t]/(t^2) \to R$ is the $R$-algebra homomorphism defined by $\varepsilon(t) = 0$. Therefore the isomorphism 1.6 can be considered as a generalization of the isomorphism* (1.2) *if the module $M$ is a $B$-algebra. Elements of the sets in the first row are specified by a pair $(\sigma, \partial)$ consisting of a homomorphism $\sigma \in \mathsf{Alg}_A(B,R)$ and an $A$-derivation $\partial \in \mathrm{Der}_A(B,R)$, where $R$ is considered as $B$-algebra via $\sigma$. In this description elements of the second row correspond to such pairs $(\sigma, \partial)$ with $\sigma = g$.*

**Remark 1.11.** *The algebra of higher differentials $\hat{\Omega}_{B/A}$ defined by Maurischat (cf.* [Mau10, Theorem 3.10]*) is a completion of $\mathrm{HS}^\infty_{B/A}$.*

---

[2] Vojta defines $\mathrm{HS}^m_{B/A}$ as the quotient of $B[b^{(i)} \mid b \in B, i \in \{1, \dots, m\}]$ by an ideal defined similarly as the ideal $I$ below, but without the relation $b^{(0)} - b$ for $b \in B$ and identifies $b^{(0)}$ with $b$. This is equivalent to the definition here.

### 1.5. Rosen's higher differentials.

**Notation:** *Let $A$ be a commutative ring.*

**Definition 1.12** ([Ros08, Definition 1.4])**.** *$A$* commutative $\mathcal{D}$-ring over *$A$ is a commutative $A$-algebra $B$ together with a higher derivation $\delta_B = (\delta_B^{(i)})_{i\in\mathbb{N}}$ of length $\infty$ from $B$ to itself such that $\delta_B^{(0)} = \mathrm{id}_B$. A commutative $\mathcal{D}$-ring $(B, \delta_B)$ over $A$ is said to be* iterative*, if the higher derivation $\delta_B$ is iterative.*

**Definition 1.13** ([Ros08, Definition 1.8])**.** *Let $(A, \delta_A)$ be a commutative $\mathcal{D}$-ring (over $A$). A commutative $A$-algebra $f\colon A \to B$ is a* commutative $(A, \delta_A)$-algebra *if for all $a \in A$, the equality $f(a) = 0$ implies $f(\delta_A^{(i)}(a)) = 0$ for all $i \in \mathbb{N}$, i.e. if $\operatorname{Ker} f$ is a $\mathcal{D}$-ideal.*

*If $f\colon A \to B_1$ and $g\colon A \to B_2$ are commutative $(A, \delta_A)$-algebras, then a* higher derivation from $B_1$ to $B_2$ of length $m$ over $(A, \delta_A)$ *is a sequence $\partial = (\partial^{(i)})_{i=0,\dots,m}$ of maps $\partial^{(i)}\colon B_1 \to B_2$ such that*

*(1) $\partial^{(i)}(b + b') = \partial^{(i)}(b) + \partial^{(i)}(b')$,*
*(2) $\partial^{(i)}(b \cdot b') = \sum_{i_1+i_2=i} \partial^{(i_1)}(b) \cdot \partial^{(i_2)}(b')$ and*
*(3) $\partial^{(i)}(f(a)) = g(\delta_A^{(i)}(a))$*

*for all $a \in A$, $b, b' \in B$ and $i = 0, \dots, m$. We denote the set of higher derivations from $B_1$ to $B_2$ of length $m$ over $(A, \delta_A)$ by $\mathrm{Der}^m_{(A,\delta_A)}(B_1, B_2)$.*

**Remark 1.14.** *If the higher derivation $\delta_A = (\delta_A)_{i\in\mathbb{N}}$ on $A$ is trivial, i.e. if $\delta_A^{(i)} = 0$ holds for all $i \geq 1$, then definition 1.13 reduces to definition 1.4.*

**Definition 1.15** ([Ros08, Definition 1.9])**.** *Given a commutative $\mathcal{D}$-ring $(A, \delta_A)$ and a commutative $(A, \delta_A)$-algebra $f\colon A \to B$, we define a $B$-algebra*

$$\mathrm{HS}^m_{B/(A,\delta_A)} := B[b^{(i)} \mid b \in B, 0 \leq i \leq m]/I,$$

*where $I$ is the ideal generated by*

$$(b + b')^{(i)} - b^{(i)} - b'^{(i)}, \quad (bb')^{(i)} - \sum_{i_1+i_2=i} b^{(i_1)} b'^{(i_2)}, \quad f(a)^{(i)} - f(\delta_A^{(i)}(a)) \quad \text{and} \quad b^{(0)} - b$$

*for all $a \in A$, all $b, b' \in B$ and all $i = 0, \dots, m$.*

**Remark 1.16.** *(1) If $(A, \delta_A)$ is the trivial higher differential ring (i.e. $\delta_A^{(0)} = \mathrm{id}_A$ and $\delta_A^{(j)} = 0$ for all $j \geq 1$), then every commutative $A$-algebra $B$ is an $(A, \delta_A)$-algebra and $\mathrm{HS}^m_{B/(A,\delta_A)}$ in definition 1.15 coincides with the $B$-algebra $\mathrm{HS}^m_{B/A}$ as defined by Vojta, cf. definition 1.7.*

*(2) If $\delta_A = (\mathrm{id}_A, \delta_A^{(1)})$ is the higher derivation of length $1$ on $A$ that is induced by a derivation $\delta_A^{(1)}\colon A \to A$ and if $f\colon A \to B$ is a commutative $A$-algebra, then there is a bijection of $B$-algebras*

$$\mathrm{HS}^1_{B/(A,\delta_A)} \cong \mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A^{(1)}}, \tag{1.7}$$

*cf. subsection 1.2.*

**Definition 1.17** ([Ros08, Definition 1.11])**.** *Let $(A, \delta_A)$ be a commutative $\mathcal{D}$-ring. Then a* commutative $\mathcal{D}$-$(A, \delta_A)$-algebra *is a commutative $\mathcal{D}$-ring $(B, \partial)$ that is also an $(A, \delta_A)$-algebra via $f\colon A \to B$ such that $\partial^{(i)}(f(a)) = f(\delta_A^{(i)}(a))$ for all $i \in \mathbb{N}$ and all $a \in A$.*

**Definition 1.18.** *Given a commutative $\mathcal{D}$-ring $(A, \delta_A)$ and a commutative $(A, \delta_A)$-algebra $f\colon A \to B$, the $B$-algebras $(\mathrm{HS}^m_{B/(A,\delta_A)})_{m\in\mathbb{N}}$ form a directed system. We denote the direct limit by*

$$\mathrm{HS}^\infty_{B/(A,\delta_A)} := \varinjlim_{m\in\mathbb{N}} \mathrm{HS}^m_{B/(A,\delta_A)}\,. \tag{1.8}$$

*For $m \in \mathbb{N}$ we define rings*

$$A_m := A[t]/(t^{m+1}) \qquad \text{and} \qquad A_\infty := A[\![t]\!] \tag{1.9}$$

*and homomorphisms*

$$e\colon A \to A_m, \quad a \mapsto \sum_{i=0}^{m} \delta_A^{(i)}(a)t^i \tag{1.10}$$

*for all $m \in \mathbb{N}$ and for $m = \infty$. We denote by $\tilde{A}_m$ the ring $A_m$ considered as an $A$-algebra via the homomorphism $e$. Similarly, we denote by $\tilde{B}_m$ the ring $B_m$ that is made into an $A$-algebra via the composition*

$$A \xrightarrow{e} A_m \to B_m, \quad a \mapsto \sum_{i=0}^{m} f(\delta_A^{(i)}(a))t^i$$

*The $A$-algebra $\tilde{B}_\infty$ is also denoted by $\widetilde{B[\![t]\!]}$.*

**Proposition 1.19** ([Ros08, Propositions 1.18 and 1.19 and Corollary 1.20])**.** *Given a commutative $\mathcal{D}$-ring $(A, \delta_A)$ and two commutative $(A, \delta_A)$-algebras $B$ and $R$, there are isomorphisms*

$$\mathrm{Der}^m_{(A,\delta_A)}(B, R) \xrightarrow{\sim} \mathsf{Alg}_A(\mathrm{HS}^m_{B/(A,\delta_A)}, R)$$

*and*

$$\mathsf{Alg}_A(B, \tilde{R}_m) \xrightarrow{\sim} \mathrm{Der}^m_{(A,\delta_A)}(B, R) \tag{1.11}$$

*and therefore also*

$$\mathsf{Alg}_A(B, \tilde{R}_m) \xrightarrow{\sim} \mathsf{Alg}_A(\mathrm{HS}^m_{B/(A,\delta_A)}, R). \tag{1.12}$$

**Remark 1.20.** *(1) The isomorphism* (1.12) *generalizes* (1.6)*.*

*(2) The diagram in remark 1.10 generalizes as follows: Let $\delta_A^{(1)}\colon A \to A$ be a derivation, $\delta_A = (\mathrm{id}_A, \delta_A^{(1)})$ be the associated higher derivation of length 1 on $A$, $f\colon A \to B$ be a commutative $(A, \delta_A)$-algebra and $g\colon B \to R$ be a commutative $B$-algebra.*

$$\begin{tikzcd}
\mathrm{Der}^1_{(A,\delta_A)}(B,R) \arrow[r, "(1.11)"] & \mathsf{Alg}_A(B,\tilde{R}_1) \arrow[r, "(1.12)"] & \mathsf{Alg}_A(\mathrm{HS}^1_{B/(A,\delta_A)},R) \arrow[r, "(1.7)"] & \mathsf{Alg}_A(\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A^{(1)}},R) \\
\mathrm{Der}_{\delta_A^{(1)}}(B,R) \arrow[u, hook] \arrow[r, "\sim"] \arrow[rrr, bend right, "(1.3)"'] & \{\theta \in \mathsf{Alg}_A(B,\tilde{R}_1) \mid \varepsilon\circ\theta = g\} \arrow[u, hook] \arrow[r, "\sim"] & \mathsf{Alg}_B(\mathrm{HS}^1_{B/(A,\delta_A)},R) \arrow[u, hook] \arrow[r, "(1.7)"] & \mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A^{(1)}},R) \arrow[u, hook]
\end{tikzcd}$$

*(3) An analogue of the isomorphism* (1.12) *exits also for $m = \infty$ and has the form*

$$\mathsf{Alg}_A(B, \widetilde{R[\![t]\!]}) \xrightarrow{\sim} \mathsf{Alg}_A(\mathrm{HS}^\infty_{B/(A,\delta_A)}, R). \tag{1.13}$$

**Lemma 1.21** ([Ros08, Lemma 1.12])**.** *If $(A, \delta_A)$ is a commutative unital iterative differential ring, then the $B$-algebra $\mathrm{HS}^\infty_{B/(A,\delta_A)}$ carries a canonical unital iterative higher derivation $d = (d^{(l)})_{l\in\mathbb{N}}$ of length $\infty$ defined by $d^{(l)}(b^{(k)}) := \binom{k+l}{k} b^{(k+l)}$ for all $k, l \in \mathbb{N}$ and all $b \in B$.*

**Proposition 1.22** ([Ros08, Proposition 1.21])**.** *Let $(A, \delta_A)$ be an iterative $\mathcal{D}$-ring. Then the forgetful functor from the category of commutative iterative $\mathcal{D}$-$(A, \delta_A)$-algebras to the category of commutative $(A, \delta_A)$-algebras has a left adjoint, that sends an $(A, \delta_A)$-algebra $B$ to $(\mathrm{HS}^\infty_{B/(A,\delta_A)}, d)$, where $d$ is the iterative derivation on $\mathrm{HS}^\infty_{B/(A,\delta_A)}$ defined in lemma 1.21.*

*Sketch of proof.* If $B$ is a commutative $(A, \delta_A)$-algebra, $(S, \delta_S)$ a commutative $\mathcal{D}$-$(A, \delta_A)$-algebra and $g\colon B \to S$ is a homomorphism of $A$-algebras, then we define $G\colon (\mathrm{HS}^\infty_{B/(A,\delta_A)}, d) \to (S, \delta_S)$ by $G(b^{(i)}) := \delta_S^{(i)}(g(b))$ for all $b \in B$ and $i \in \mathbb{N}$. ☐

1.6. **Differential prolongations.** Gillet constructs prolongations in section 1.2 of [Gil02]. We summarize his results.

**Notation:** *Let $(A, \delta_A)$ be a commutative differential ring.*

**Definition 1.23.** *We denote by* $\mathsf{DifferentialCAlg}_{(A,\delta_A)}$ *the category of* commutative differential $(A, \delta_A)$-algebras*, the objects being commutative $A$-algebras $f\colon A \to B$ equipped with a derivation $\delta_B\colon B \to B$ such that $\delta_B \circ f = f \circ \delta_A$, and the morphisms from a commutative differential $(A, \delta_A)$-algebra $(B_1, \delta_{B_1})$ to $(B_2, \delta_{B_2})$ being morphisms of $A$-algebras $\varphi\colon B_1 \to B_2$ such that $\varphi \circ \delta_{B_1} = \delta_{B_2} \circ \varphi$.*

**Proposition 1.24** ([Gil02, Proposition 1.19])**.** *The forgetful functor*

$$U\colon \mathsf{DifferentialCAlg}_{(A,\delta_A)} \to \mathsf{CAlg}_A, \quad (B, \delta_B) \mapsto B$$

*has a left adjoint*

$$(\ )^\infty\colon \mathsf{CAlg}_A \to \mathsf{DifferentialCAlg}_{(A,\delta_A)}, \quad B \mapsto (B^\infty, \delta),$$

*i.e. for every commutative $A$-algebra $B$ and every commutative differential $(A, \delta_A)$-algebra $(S, \delta_S)$ there is a bijection*

$$\mathsf{DifferentialCAlg}_{(A,\delta_A)}((B^\infty, \delta), (S, \delta_S)) \cong \mathsf{CAlg}_A(B, S). \tag{1.14}$$

*Proof.* For a proof using general properties of forgetful functors we refer to [Gil02, Proposition 1.19]. We will construct the algebra $B^\infty$ in proposition 1.29 below and show that it has the required property. ☐

**Lemma 1.25** ([Gil02, Lemma 1.21])**.** *Let $X$ be a set. Then the functor*

$$\mathsf{DifferentialCAlg}_{(A,\delta_A)} \to \mathsf{Set}, \quad (B, \delta_B) \mapsto B^X (:= \mathsf{Set}(X, B))$$

*is representable, i.e. there is a commutative differential $(A, \delta_A)$-algebra $(A\{X\}, \delta)$ such that*

$$\mathsf{DifferentialCAlg}_{(A,\delta_A)}((A\{X\}, \delta), (B, \delta_B)) \cong B^X$$

*for all commutative differential $(A, \delta_A)$-algebras $(B, \delta_B)$. In other words the forgetful functor from the category* $\mathsf{DifferentialCAlg}_{(A,\delta_A)}$ *to* $\mathsf{Set}$ *is representable.*

*Proof.* We define $A\{X\}$ to be the differential ring over $(A, \delta_A)$ in the variables $X$, which is defined as the polynomial $A$-algebra in the variables $(x^{(i)})_{x\in X, i\in\mathbb{N}}$ equipped with the derivation $\delta$ extending $\delta_A$ by $\delta(x^{(i)}) := x^{(i+1)}$ for all $x \in X$ and $i \in \mathbb{N}$. Given a function $f\colon X \to B$, we associate to it the function $\bar{f}\colon A\{X\} \to B$ defined by sending $x^{(i)}$ to $\delta_B^i(f(x))$ for all $x \in X$ and $i \in \mathbb{N}$. ☐

**Definition 1.26.** *The differential ring $A\{X\}$ is called the* ring of differential polynomials on the set $X$ over $(A, \delta_A)$.

**Lemma 1.27** ([Gil02, Lemma 1.23])**.** *Let $(A, \delta_A)$ be a differential field and $X$ be a set. Then $(A[X])^\infty$ is isomorphic to the differential polynomial ring $A\{X\}$.*

*Proof.* The left adjoint of the forgetful functor $\mathsf{CAlg}_A \to \mathsf{Set}$ is given by $X \mapsto A[X]$. The left adjoint of the forgetful functor $\mathsf{DifferentialCAlg}_{(A,\delta_A)} \to \mathsf{CAlg}_A$ is given by $B \to (B^\infty, \delta)$ and the left adjoint of the forgetful functor $\mathsf{DifferentialCAlg}_{(A,\delta_A)} \to \mathsf{Set}$ is

$$\mathsf{Set} \to \mathsf{DifferentialCAlg}_{(A,\delta_A)}, \quad X \mapsto (A\{X\}, \delta).$$

The composition of the first and second adjoint functor is the third one. Therefore we obtain $(A[X])^\infty \cong A\{X\}$. ☐

Let $(A, \delta_A)$ be a commutative differential ring and $f\colon A \to B$ be a commutative $A$-algebra. We define $B^{(-1)} := A$, $\rho_{-1} := f$ and $\delta^{(-1)} := \rho_{-1} \circ \delta_A$. We define a category $\mathsf{T}_{(A,\delta_A,B)}$ as follows: The objects are the sequences $(B^{(i)}, \rho_i, \delta^{(i)})_{i\in\mathbb{N}}$ consisting of

1. commutative rings $B^{(i)}$,
2. ring homomorphism $\rho_i\colon B^{(i)} \to B^{(i+1)}$ and
3. derivations $\delta^{(i)}\colon B^{(i)} \to B^{(i+1)}$, where we consider $B^{(i+1)}$ as $B^{(i)}$-algebra via $\rho_i$

such that $B^{(0)} = B$ and

$$\rho_i \circ \delta^{(i-1)} = \delta^{(i)} \circ \rho_{i-1} \tag{1.15}$$

for each $i \geq 0$. The morphisms from $(B^{(i)}, \rho_i, \delta^{(i)})_{i\in\mathbb{N}}$ to $(B'^{(i)}, \rho'_i, \delta'^{(i)})_{i\in\mathbb{N}}$ are families of ring homomorphisms $(f_i \colon B^{(i)} \to B'^{(i)})_{i\in\mathbb{N}}$ such that $f_0 = \mathrm{id}_B$,

$$\delta'^{(i)} \circ f_i = f_{i+1} \circ \delta^{(i)} \qquad \text{and} \qquad \rho'_i \circ f_i = f_{i+1} \circ \rho_i \tag{1.16}$$

for all $i \in \mathbb{N}$.

**Remark 1.28.** *Let $(A, \delta_A)$ be a commutative differential ring and $f \colon A \to B$ be a commutative $A$-algebra. Let further $(S, \delta_S)$ be a commutative differential $(A, \delta_A)$-algebra that is a $B$-algebra via $g \colon B \to S$. To this data there is associated naturally an element of $\mathsf{T}_{(A,\delta_A,B)}$ given by*

$$\begin{aligned}
B^{(-1)} &:= A, & B^{(0)} &:= B & B^{(i)} &:= S\\
\rho_{-1} &:= f, & \rho_0 &:= g & \rho_i &:= \mathrm{id}_S\\
\delta^{(-1)} &:= f \circ \delta_A, & \delta^{(0)} &:= \delta_S \circ g & \delta^{(i)} &:= \delta_S
\end{aligned}$$

*for all $i \geq 1$.*

**Proposition 1.29** ([Gil02, Proposition 1.26])**.** *Let $(A, \delta_A)$ be a commutative differential ring and $B$ be a commutative $A$-algebra via $f \colon A \to B$.*

*(1) The category $\mathsf{T}_{(A,\delta_A,B)}$ has an initial object $(B^{(i)}, \rho_i, \delta^{(i)})_{i\in\mathbb{N}}$.*

*(2) The algebras $(B^{(i)})_{i\in\mathbb{N}\cup\{-1\}}$ form a direct system via the homomorphisms $(\rho_i \colon B^{(i)} \to B^{(i+1)})_{i\in\mathbb{N}\cup\{-1\}}$ and the derivations $\delta^{(i)} \colon B^{(i)} \to B^{(i+1)}$ induce a derivation $\delta$ on the direct limit*

$$B^\infty := \varinjlim_{i\in\mathbb{N}\cup\{-1\}} B^{(i)}.$$

*(3) For every commutative differential $(A, \delta_A)$-algebra $(S, \delta_S)$ and every $n \in \mathbb{N}$ there are natural bijections between the following sets:*

$$\mathsf{CAlg}_A(B, S)$$

$$\{(f^{(i)})_{i=0,\dots,n} \in \varprojlim_{i=0,\dots,n} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\}$$

$$\{(f^{(i)})_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\}$$

$$\mathsf{DifferentialCAlg}_{(A,\delta_A)}((B^\infty, \delta), (S, \delta_S))$$

*Proof.* We construct the sequence $(B^{(n)}, \rho_n, \delta^{(n)})_{n\in\mathbb{N}\cup\{-1\}}$ by induction on $n$. We have by definition $B^{(-1)} = A$, $B^{(0)} = B$, $\rho_{-1} = f$ and $\delta^{(-1)} = f \circ \delta_A$. Assume that for some $n \geq 1$ a sequence

$$\begin{tikzcd}
B^{(-1)} \arrow[r, shift left, "\delta^{(-1)}"] \arrow[r, shift right, "\rho_{-1}"'] & B^{(0)} \arrow[r, shift left, "\delta^{(0)}"] \arrow[r, shift right, "\rho_0"'] & {} & \cdots & {} \arrow[r, shift left, "\delta^{(n-2)}"] \arrow[r, shift right, "\rho_{n-2}"'] & B^{(n-1)}
\end{tikzcd}$$

is given such that the relations (1.15) hold for all $i = 0, \dots, n-2$. We define $B^{(n)}$ to be $\mathrm{Sym}_{B^{(n-1)}}(\Omega^1_{B^{(n-1)}/\mathbb{Z}})/I_{\delta^{(n-2)}}$, the morphism $\rho_{n-1}$ to be the natural $B^{(n-1)}$-algebra structure of $\mathrm{Sym}_{B^{(n-1)}}(\Omega^1_{B^{(n-1)}/\mathbb{Z}})/I_{\delta^{(n-2)}}$ and $\delta^{(n-1)}$ to be the derivation $\mathrm{d} \colon B^{(n-1)} \to \mathrm{Sym}_{B^{(n-1)}}(\Omega^1_{B^{(n-1)}/\mathbb{Z}})/I_{\delta^{(n-2)}}$ extending $\delta^{(n-2)}$ that we obtain by applying subsection 1.2 to $B^{(n-2)} \underset{\rho_{n-2}}{\overset{\delta^{(n-2)}}{\rightrightarrows}} B^{(n-1)}$ .

In order to see that this sequence is an initial object, let $(B'^{(n)}, \rho'_n, \delta'^{(n)})_{n\in\mathbb{N}}$ be an arbitrary object in $\mathsf{T}_{(A,\delta_A,B)}$. We define a morphism $(B^{(n)}, \rho_n, \delta^{(n)})_{n\in\mathbb{N}} \to (B'^{(n)}, \rho'_n, \delta'^{(n)})_{n\in\mathbb{N}}$ inductively as follows: First we define $f_0 := \mathrm{id}_{B^{(0)}}$. Assume the morphisms $f_0, \dots, f_{n-1}$ are already defined for some $n \geq 1$ and fulfill the commutation relations (1.16) for all $i = 0, \dots, n-2$. We consider $B'^{(n)}$ as $B^{(n-1)}$-algebra via $\rho'_{n-1} \circ f_{n-1}$. The derivation $\delta'^{(n-1)} \circ f_{n-1}$ extends $\delta^{(n-2)}$, since

$$\rho'_{n-1} \circ f_{n-1} \circ \delta^{(n-2)} = \rho'_{n-1} \circ \delta'^{(n-2)} \circ f_{n-2} = \delta'^{(n-1)} \circ \rho'_{n-2} \circ f_{n-2} = \delta'^{(n-1)} \circ f_{n-1} \circ \rho_{n-2}.$$

Therefore by lemma 1.2 there exists a unique $f_n \in \mathsf{Alg}_{B^{(n-1)}}(B^{(n)}, B'^{(n)})$ such that $\delta'^{(n-1)} \circ f_{n-1} = f_n \circ \delta^{(n-1)}$. Since $f_n$ is a morphism of $B^{(n-1)}$-algebras, we conclude that $f_n \circ \rho_{n-1} = \rho'_{n-1} \circ f_{n-1}$.

$$\begin{tikzcd}
B^{(n-2)} \arrow[r, shift left, "\delta^{(n-2)}"] \arrow[r, shift right, "\rho_{n-2}"'] \arrow[d, "f_{n-2}"] & B^{(n-1)} \arrow[r, shift left, "\delta^{(n-1)}"] \arrow[r, shift right, "\rho_{n-1}"'] \arrow[d, "f_{n-1}"] & B^{(n)} \arrow[d, dotted, "f_n"] \\
B'^{(n-2)} \arrow[r, shift left, "\delta'^{(n-2)}"] \arrow[r, shift right, "\rho'_{n-2}"'] & B'^{(n-1)} \arrow[r, shift left, "\delta'^{(n-1)}"] \arrow[r, shift right, "\rho'_{n-1}"'] & B'^{(n)}
\end{tikzcd}$$

By induction we obtain a morphism $(f_n)_{n\in\mathbb{N}}$ from $(B^{(n)}, \rho_n, \delta^{(n)})_{n\in\mathbb{N}}$ to $(B'^{(n)}, \rho'_n, \delta'^{(n)})_{n\in\mathbb{N}}$.

The affirmation in part (2) is clear.

To prove that for every commutative differential $(A, \delta_A)$-algebra $(S, \delta_S)$ there is a bijection

$$\mathsf{DifferentialCAlg}_{(A,\delta_A)}((B^\infty, \delta), (S, \delta_S)) \cong \mathsf{CAlg}_A(B, S)$$

we note the following: Let $g\colon B \to S$ be a homomorphism of $A$-algebras. By remark 1.28 the sequence

$$\begin{tikzcd}
A \arrow[r, shift left, "f\circ\delta_A"] \arrow[r, shift right, "f"'] & B \arrow[r, shift left, "\delta_S\circ g"] \arrow[r, shift right, "g"'] & S \arrow[r, shift left, "\delta_S"] \arrow[r, shift right, "\mathrm{id}_S"'] & S \arrow[r, shift left, "\delta_S"] \arrow[r, shift right, "\mathrm{id}_S"'] & S & \dots
\end{tikzcd}$$

is an element of the category $\mathsf{T}_{(A,\delta_A,B)}$. Using the universal property of the initial object, we obtain homomorphisms $(f_i)_{i\in\mathbb{N}}$ such that the following diagrams commute[3]

$$\begin{tikzcd}
B^{(-1)} \arrow[r, shift left, "\delta^{(-1)}"] \arrow[r, shift right, "\rho_{-1}"'] \arrow[d, "\mathrm{id}_A"] & B^{(0)} \arrow[r, shift left, "\delta^{(0)}"] \arrow[r, shift right, "\rho_0"'] \arrow[d, "\mathrm{id}_B"] & B^{(1)} \arrow[r, shift left, "\delta^{(1)}"] \arrow[r, shift right, "\rho_1"'] \arrow[d, "f_1"] & B^{(2)} \arrow[r, shift left, "\delta^{(2)}"] \arrow[r, shift right, "\rho_2"'] \arrow[d, "f_2"] & \dots \\
A \arrow[r, shift left, "f\circ\delta_A"] \arrow[r, shift right, "f"'] & B \arrow[r, shift left, "\delta_S\circ g"] \arrow[r, shift right, "g"'] & S \arrow[r, shift left, "\delta_S"] \arrow[r, shift right, "\mathrm{id}_S"'] & S \arrow[r, shift left, "\delta_S"] \arrow[r, shift right, "\mathrm{id}_S"'] & \dots,
\end{tikzcd}$$

where $B^{(-1)} = A$, $B^{(0)} = B$, $\rho_{-1} = f$ and $\delta^{(-1)} = f \circ \delta_A$. Together with $f_0 \coloneqq \mathrm{id}_B$ we obtain an element $(f_i)_{i\in\mathbb{N}} \in \{(f^{(i)})_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\}$ and therefore a map

$$\mathsf{CAlg}_A(B, S) \to \{(f^{(i)})_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\},$$

which is bijective with inverse given by the map that sends $(f_i)_{i\in\mathbb{N}}$ to $f_0$. By truncation we obtain maps from

$$\{(f^{(i)})_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\}$$

to

$$\{(f^{(i)})_{i=0}^{n} \in \varprojlim_{i=0,\dots,n} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\}$$

for all $n \in \mathbb{N}$, which are also bijective as we have seen in the proof of part (1). Finally we note that the bijection $\varprojlim_{i\in\mathbb{N}} \mathsf{CAlg}_A(B^{(i)}, S) \cong \mathsf{CAlg}_A(B^\infty, S)$ restricts to a bijection between $\{(f^{(i)})_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} \mathsf{CAlg}_A(B^{(i)}, S) \mid f^{(i)} \circ \delta^{(i-1)} = \delta_S \circ f^{(i-1)}\}$ and $\mathsf{DifferentialCAlg}_{(A,\delta_A)}((B^\infty, \delta), (S, \delta_S))$. ☐

**Example 1.30.** *Let* $B = A[x]$ *be the polynomial ring over* $A$*, considered as* $A$*-algebra via the homomorphism* $f\colon A \to A[x]$ *that sends* $a \in A$ *to the constant polynomial* $a$*. If we apply proposition 1.29 in this situation, then we obtain* $B^{(1)} = \mathrm{Sym}_{A[x]}(\Omega_{A[x]/\mathbb{Z}})/I_{\partial_A} = A[x^{(0)}, x^{(1)}]$*, where we denote* $x$ *by* $x^{(0)}$ *and* $\mathrm{d}\,x$ *by* $x^{(1)}$ *and the derivation* $\delta^{(0)}\colon A[x] \to A[x^{(0)}, x^{(1)}]$ *is the derivation extending* $f \circ \partial_A$ *by* $\delta^{(0)}(x) = x^{(1)}$*. In general* $B^{(n)} =$

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[3] We consider this as two diagrams, one formed by the upper one of each pair of horizontal arrows and one with the lower one.

$\mathrm{Sym}_{A[x^{(0)},x^{(1)},\ldots,x^{(n-1)}]}(\Omega^1_{A[x^{(0)},x^{(1)},\ldots,x^{(n-1)}]/\mathbb{Z}})/I_{\delta^{(n-2)}} = A[x^{(0)},x^{(1)},\ldots,x^{(n)}]$, *and* $B^\infty = A[x^{(n)} \mid n \in \mathbb{N}]$ *with derivation* $\delta\colon B^\infty \to B^\infty$ *extending* $\delta_A$ *by* $\delta(x^{(i)}) = x^{(i+1)}$ *for all* $i \in \mathbb{N}$*. The differential ring* $B^\infty$ *is usually denoted by* $A\{x\}$ *and is called the* differential polynomial algebra*.*

1.7. **Difference prolongations.** We recall a difference analogue of the differential prolongations, cf. [Hru04, p. 21], [Tom11, Proposition 2.1] or [Wib13, Proposition 1.1.26].

**Definition 1.31.** *If* $A$ *is a commutative ring and* $\sigma_A$ *an endomorphism of* $A$*, then a* commutative difference $(A,\sigma_A)$-algebra *is a commutative* $A$*-algebra* $f\colon A \to B$ *together with an endomorphism* $\sigma_B$ *of* $B$ *such that* $\sigma_B \circ f = f \circ \sigma_A$*.*

*Let* $\mathsf{DifferenceCAlg}_{(A,\sigma_A)}$ *denote the category of commutative difference* $(A,\sigma_A)$*-algebras, the morphisms from* $(B_1,\sigma_{B_1})$ *to* $(B_2,\sigma_{B_2})$ *being morphisms of* $A$*-algebras* $\varphi\colon B_1 \to B_2$ *such that* $\varphi \circ \sigma_{B_1} = \sigma_{B_2} \circ \varphi$*.*

**Proposition 1.32.** *Let* $A$ *be a field and* $\sigma_A$ *be an endomorphism of the field* $A$*. Let further* $B$ *be a commutative* $A$*-algebra. Then there exists a commutative difference* $(A,\sigma_A)$*-algebra* $([\sigma]_A B,\sigma)$ *and a morphism* $\psi\colon B \to [\sigma]_A B$ *of* $A$*-algebras satisfying the following universal property: For every commutative difference* $(A,\sigma_A)$*-algebra* $(S,\sigma_S)$ *and every morphism* $g\colon B \to S$ *of* $A$*-algebras there exists a unique morphism* $G\colon [\sigma]_A B \to S$ *of difference* $(A,\sigma_A)$*-algebras such that the diagram*

$$\begin{tikzcd} B \arrow[rr, "\psi"] \arrow[dr, "g"'] & & {[\sigma]_A B} \arrow[dl, "G"] \\ & S & \end{tikzcd} \tag{1.17}$$

*commutes. The commutative difference* $(A,\sigma_A)$*-algebra* $([\sigma]_A B,\sigma)$ *is unique up to unique isomorphism in the sense that for every commutative difference* $(A,\sigma_A)$*-algebra* $(S,\sigma_S)$ *there is a bijection*

$$\mathsf{CAlg}_A(B,S) \xrightarrow{\sim} \mathsf{DifferenceCAlg}_{(A,\sigma_A)}(([\sigma]_A B,\sigma),(S,\sigma_S)),$$

*i.e. the forgetful functor* $\mathsf{DifferenceCAlg}_{(A,\sigma_A)} \to \mathsf{CAlg}_A$ *has a left-adjoint.*

*Proof.* For $i \in \mathbb{N}$ let ${}^{\sigma_A^i}B$ be the ring $B \otimes_A A$, where the $A$-algebra structure on the right factor is $\sigma_A^i$. We consider ${}^{\sigma_A^i}B$ as $A$-algebra via the right factor, which is considered as $A$-algebra via the identity on $A$. There is a homomorphism of rings

$$\psi_i\colon {}^{\sigma_A^i}B \to {}^{\sigma_A^{i+1}}B, \quad b \otimes a \mapsto b \otimes \sigma_A(a).$$

We define

$$B_i \coloneqq B \otimes_A {}^{\sigma_A}B \otimes_A \cdots \otimes_A {}^{\sigma_A^i}B. \tag{1.18}$$

The family of $A$-algebras $(B_i)_{i\in\mathbb{N}}$ becomes a direct system via the morphisms

$$B_i \to B_{i+1}, \quad b_0 \otimes \cdots \otimes b_i \mapsto b_0 \otimes \cdots \otimes b_i \otimes 1$$

and we define $[\sigma]_A B$ as the direct limit $\varinjlim_{i\in\mathbb{N}} B_i$.

The morphisms

$$\sigma_i\colon R_i \to R_{i+1}, \quad b_0 \otimes \cdots \otimes b_i \mapsto 1 \otimes \psi_0(b_0) \otimes \cdots \otimes \psi_i(b_i)$$

induce a morphism $\sigma\colon [\sigma]_A B \to [\sigma]_A B$. We define $\psi\colon B \to [\sigma]_A B$ by identifying $B$ with $B_0$.

Let $(S,\sigma_S)$ be a commutative difference $(A,\sigma_A)$-algebra and let $g\colon B \to S$ be a morphism of $A$-algebras. We define morphisms of $A$-algebras

$$G_i\colon {}^{\sigma_A^i}B \to S, \quad b \otimes a \mapsto \sigma_S^i(g(b)) \cdot g(f(a)).$$

The morphisms $G_i$ induce morphisms of $A$-algebras

$$B_k \to S, \quad b_0 \otimes \cdots \otimes b_k \mapsto G_0(b_0) \ldots G_k(b_k)$$

for all $k \in \mathbb{N}$ and finally a morphism of $A$-algebras

$$G\colon [\sigma]_A B \to S.$$

For $b \otimes a \in {}^{\sigma^i}B$ we have

$$G_{i+i}(\psi_i(b\otimes a))=G_{i+1}(b\otimes\sigma_A(a))=\sigma_S^{i+1}(g(b))g(f(\sigma_A(a)))=\sigma_S^{i+1}(g(b))\sigma_S(g(f(a)))=\sigma_S(\sigma_S^{i}(g(b))g(f(a)))=\sigma_S(G_i(b\otimes a))$$

and therefore

$$\begin{aligned} G(\sigma(b_0 \otimes \cdots \otimes b_i)) &= G(1 \otimes \psi_0(b_0) \otimes \cdots \otimes \psi_i(b_i)) \\ &= G_1(\psi_0(b_0)) \ldots G_{i+1}(\psi_i(b_i)) \\ &= \sigma_S(G_0(b_0)) \ldots \sigma_S(G_i(b_i)) \\ &= \sigma_S(G(b_0 \otimes \cdots \otimes b_i)), \end{aligned}$$

i.e. $G$ is a morphism of difference $(A, \sigma_A)$-algebras. The diagram (1.17) commutes, since we have $G(\psi(b)) = G(b) = G_0(b) = g(b)$ for all $b \in B$. In order to show that the morphism $G$ is unique, it is enough to show that any morphism $G' \colon [\sigma]_A B \to S$ of difference $(A, \sigma_A)$-algebras coincides with $G$ on each $B_i$, $i \in \mathbb{N}$, and it is even enough to show that it coincides with $G_i$ on each ${}^{\sigma_A^i}B$, which is the case, since $G'(\sigma^i(\psi(b))) = \sigma_S^i(G'(\psi(b))) = \sigma_S^i(g(b)) = G_i(\sigma^i(\psi(b)))$. The uniqueness of $[\sigma]_A B$ can be shown as usual. ☐

**Example 1.33.** *Let $A$ be a field with endomorphism $\sigma_A$ of $A$ and let $B = A[x]$ be the polynomial algebra in one variable $x$ over $A$. Then the difference $(A, \sigma_A)$-algebra $[\sigma]_A B$ in proposition 1.32 is isomorphic to the difference polynomial ring*

$$A\{x\} := A[\sigma^n(x) \mid n \in \mathbb{N}]$$

*equipped with the endomorphism $\sigma$ extending $\sigma_A$ via*

$$\sigma \colon A\{x\} \to A\{x\}, \quad \sigma^n(x) \mapsto \sigma^{n+1}(x)$$

*Proof.* The $A$-algebras ${}^{\sigma_A^i}B$ in the proof of proposition 1.32 are isomorphic to $B$ itself. Therefore $B_i$ is an $(i+1)$-fold tensor product of $B = A[x]$ with itself over $A$, which is isomorphic to the polynomial algebra $A[x, \sigma(x), \ldots, \sigma^i(x)]$. Their direct limit $[\sigma]_A B$ is isomorphic to $A\{x\}$ and it is easy to see that the endomorphism $\sigma$ on $A\{x\}$ corresponds to $\sigma \colon [\sigma]_A B \to [\sigma]_A B$ under this isomorphism. ☐

1.8. **Differential kernels and their prolongations.** Johnson introduces differential kernels and their prolongations in [Joh82] and [Joh85].

**Definition 1.34** ([Joh85, §I.1, p.176] or [Joh82, I.1, p. 94])**.** *An $m$-*differential kernel *is a homomorphism of commutative rings $f \colon U_1 \to U_2$ together with derivations $\delta_1, \ldots, \delta_m$ from $U_1$ to $U_2$ (we consider $U_2$ as $U_1$-module via the $U_1$-algebra structure $f$).*

*A morphism of $m$-differential kernels from $U_1 \xrightarrow{(f,\delta_1,\ldots,\delta_m)} U_2$ to $V_1 \xrightarrow{(g,\partial_1,\ldots,\partial_m)} V_2$ consists of two ring homomorphisms $\varphi_j \colon U_j \to V_j$ for $j = 1, 2$ such that the diagrams*

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\varphi_1"] & U_2 \arrow[d, "\varphi_2"] \\ V_1 \arrow[r, "g"] & V_2 \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} U_1 \arrow[r, "\delta_i"] \arrow[d, "\varphi_1"] & U_2 \arrow[d, "\varphi_2"] \\ V_1 \arrow[r, "\partial_i"] & V_2 \end{tikzcd}$$

*commute for all $i \in \{1, \ldots, m\}$.*

**Definition 1.35** ([Joh82, p. 95])**.** *Let $U \xrightarrow{(f,\delta_1,\ldots,\delta_m)} V$ and $V \xrightarrow{(g,\partial_1,\ldots,\partial_m)} W$ be two $m$-differential kernels. We say that $(g, \partial_1, \ldots, \partial_m)$* prolongs *$(f, \delta_1, \ldots, \delta_m)$ if we have for all $u \in U$ that $\partial_i(f(u)) = g(\delta_i(u))$ and $\partial_i(\delta_j(u)) = \partial_j(\delta_i(u))$, i.e. if the diagrams*

$$\begin{tikzcd} U \arrow[r, "\delta_i"] \arrow[d, "f"] & V \arrow[d, "g"] \\ V \arrow[r, "\partial_i"] & W \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} U \arrow[r, "\delta_i"] \arrow[d, "\delta_j"] & V \arrow[d, "\partial_j"] \\ V \arrow[r, "\partial_i"] & W \end{tikzcd}$$

*commute for all $i, j \in \{1, \ldots, m\}$.*

*If* $V \xrightarrow{(g,\partial_1,\dots,\partial_m)} W$ *and* $V \xrightarrow{(g',\partial'_1,\dots,\partial'_m)} W'$ *are prolongations of the differential kernel* $U \xrightarrow{(f,\delta_1,\dots,\delta_m)} V$*, then a* morphism *from* $(g,\partial_1,\dots,\partial_m)$ *to* $(g',\partial'_1,\dots,\partial'_m)$ *is any ring homomorphism* $h\colon W \to W'$ *such that* $(\mathrm{id}_V, h)$ *is a morphism of the differential kernel* $(g,\partial_1,\dots,\partial_m)$ *to* $(g',\partial'_1,\dots,\partial'_m)$*.*

*If* $V \xrightarrow{(g,\partial_1,\dots,\partial_m)} W$ *is a prolongation of the differential kernel* $U \xrightarrow{(f,\delta_1,\dots,\delta_m)} V$ *and* $h\colon W \to W'$ *is a ring homomorphism, then* $g' := h \circ g\colon V \to W'$ *together with the derivations* $\partial'_i := h \circ \partial_i$ *form a differential kernel that prolongs* $U \to V$*, since the diagrams*

$$\begin{tikzcd}
U \arrow[r, "\delta_i"] \arrow[d, "f"] & V \arrow[d, "g"] \arrow[ddr, bend left, "g'"] & \\
V \arrow[r, "\partial_i"] \arrow[drr, bend right, "g'"'] & W \arrow[dr, "h"] & \\
& & W'
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
U \arrow[r, "\delta_i"] \arrow[d, "\delta_j"] & V \arrow[d, "\partial_j"] \arrow[ddr, bend left, "\partial'_j"] & \\
V \arrow[r, "\partial_i"] \arrow[drr, bend right, "\partial'_i"'] & W \arrow[dr, "h"] & \\
& & W'
\end{tikzcd}$$

*commute. We says that this is the prolongation* defined by $h$*.*

**Proposition 1.36** (Proposition 1.4 in [Joh82])**.** *Let* $A \xrightarrow{(f,\delta)} B$ *be a differential kernel. There exists a differential kernel* $B \xrightarrow{(g,\partial)} C$ *prolonging* $(f,\delta)$ *such that if* $B \xrightarrow{(g',\partial')} C'$ *is another prolongation of* $(f,\delta)$*, then* $(g',\partial')$ *is defined by* $h$ *for a unique* $h\colon C \to C'$*.*

**Example 1.37.** *If* $B = A[x]$*, then the universal prolongation of the differential kernel that is given by the inclusion* $A \hookrightarrow A[x]$ *and the trivial derivation, which is provided by proposition 1.36, is given by the inclusion* $A[x] \hookrightarrow A[x, X^{(1)}]$ *and the derivation* $\partial\colon A[x] \to A[x, X^{(1)}]$ *over* $A$ *that is defined by* $\partial(x) = X^{(1)}$*. Note that this is the same as* $B \to B^{(1)}$ *in example 1.30.*

1.9. **Difference kernels and their prolongations.** We recall the definition of difference kernels due to Wibmer (cf. [Wib13] or [Wib12] for a more general definition).

**Notation:** *Let* $K$ *be a difference field (i.e. a field equipped with an endomorphism* $\sigma_K$*) and* $t$ *be a natural number greater or equal to* 1.

Let $K\{X\}$ be the difference polynomial ring over $(K,\sigma_K)$, cf. example 1.33. There is a family $(K[x,\sigma(x),\dots,\sigma^t(x)])_{t\in\mathbb{N}}$ of subrings of $K\{x\}$ and the endomorphism $\sigma$ of the $K$-algebra $K\{x\}$ restricts to homomorphisms of $K$-algebras $\sigma\colon K[x,\dots,\sigma^{t-1}(x)] \to K[x,\dots,\sigma^t(x)]$ such that $\sigma(\sigma^j(x)) := \sigma^{j+1}(x)$ for all $j \in \{0,\dots,t-1\}$.

**Definition 1.38.** *A* difference kernel of length $t$ *in the difference polynomial ring* $K\{x\}$ *is a prime ideal* $\mathfrak{p}_t$ *of the subring* $K[x,\dots,\sigma^t x]$ *of* $K\{x\}$ *such that* $\sigma^{-1}(\mathfrak{p}_t) = \mathfrak{p}_t \cap K[x,\dots,\sigma^{t-1}(x)]$*.*

**Definition 1.39.** *A* prolongation *of a difference kernel* $\mathfrak{p}_t$ *of length* $t$ *is a difference kernel* $\mathfrak{p}_{t+1}$ *of length* $t+1$ *such that* $\mathfrak{p}_{t+1} \cap K[x,\dots,\sigma^t x] = \mathfrak{p}_t$ *holds.*

**Remark 1.40.** *Let* $K$ *be a field and* $\mathfrak{p}_t \subseteq K[x,\dots,\sigma^t(x)]$ *be a difference kernel of length* $t$*. It induces two homomorphisms:*

*(1) The homomorphism*

$$\sigma\colon K[x,\dots,\sigma^{t-1}(x)] \to K[x,\dots,\sigma^t(x)], \quad \sigma^i(x) \mapsto \sigma^{i+1}(x)$$

*induces a ring homomorphism*

$$\bar{\sigma}\colon K[x,\dots,\sigma^{t-1}(x)]/(\mathfrak{p}_t \cap K[x,\dots,\sigma^{t-1}(x)]) \to K[x,\dots,\sigma^t(x)]/\mathfrak{p}_t.$$

*(2) The inclusion* $\iota\colon K[x,\dots,\sigma^{t-1}x] \hookrightarrow K[x,\dots,\sigma^t x]$ *induces an injection*

$$\bar{\iota}\colon K[x,\dots,\sigma^{t-1}x]/(\mathfrak{p}_t \cap K[x,\dots,\sigma^{t-1}(x)]) \to K[x,\dots,\sigma^t x]/\mathfrak{p}_t.$$

*A prolongation* $\mathfrak{p}_{t+1}$ *of* $\mathfrak{p}_t$ *also induces two homomorphisms*

$$\bar{\iota}\colon K[x,\dots,\sigma^t x]/(\mathfrak{p}_t \cap K[x,\dots,\sigma^t x]) \to K[x,\dots,\sigma^{t+1}x]/\mathfrak{p}_{t+1}.$$

*and*

$$\bar\sigma\colon K[x,\dots,\sigma^t x]/(\mathfrak{p}_t\cap K[x,\dots,\sigma^t x])\to K[x,\dots,\sigma^{t+1}x]/\mathfrak{p}_{t+1}$$

*such that the following diagram commutes*

$$\begin{tikzcd}
K[x,\dots,\sigma^{t-1}x]/(\mathfrak{p}_t\cap K[x,\dots,\sigma^{t-1}x]) \arrow[r, "\bar\iota"] \arrow[d, "\bar\sigma"] & K[x,\dots,\sigma^t x]/\mathfrak{p}_t \arrow[d, "\bar\sigma"] \\
K[x,\dots,\sigma^t x]/\mathfrak{p}_t \arrow[r, "\bar\iota"] & K[x,\dots,\sigma^{t+1}x]/\mathfrak{p}_{t+1}.
\end{tikzcd}$$

*In the form of pairs $(\bar\iota,\bar\sigma)$ a difference kernel and its prolongations are of a similar nature as* 1*-differential kernels and their prolongations as defined by Johnson (cf. subsection 1.8), just with a homomorphism $\bar\sigma$ instead of a derivation $\partial$.*

## 2. Measurings and module algebras

Given a $k$-coalgebra $D$, we recall the definition of $D$-measurings and, if $D$ is a $k$-bialgebra, of $D$-module algebras.

For $k$-modules $A, B$ and $D$ there is an isomorphism of (left) $k$-modules

$$ {}_k\mathcal{M}(D\otimes_k A, B)\to {}_k\mathcal{M}(A, {}_k\mathcal{M}(D,B)),\quad \psi\mapsto (a\mapsto (d\mapsto \psi(d\otimes a))). \tag{2.1}$$

**Lemma 2.1.** *If $(D,\Delta_D,\varepsilon_D)$ is a $k$-coalgebra and $(B,\mathrm{m}_B,\eta_B)$ is a $k$-algebra, then the $k$-module ${}_k\mathcal{M}(D,B)$ becomes a $k$-algebra with respect to the* convolution product*, defined by*

$$f\cdot g := \mathrm{m}_B\circ(f\otimes g)\circ\Delta_D$$

*for $f,g\in {}_k\mathcal{M}(D,B)$, and unit element given by the composition*

$$D\xrightarrow{\varepsilon_D} k\xrightarrow{\eta_B} B.$$

*Furthermore, $D$ is cocommutative if and only if ${}_k\mathcal{M}(D,B)$ is commutative for every commutative $k$-algebra $B$.*

*If $B$ is commutative, then ${}_k\mathcal{M}(D,B)$ is a $B$-algebra via*

$$\rho_0\colon B\to {}_k\mathcal{M}(D,B),\quad b\mapsto (d\mapsto \varepsilon(d)b).$$

*Proof.* See for example [BW03, 1.3] for a proof of the first two statements. The last statement holds, since for all $d\in D$ and all $b\in B$ we have

$$(\rho_0(b)\cdot f)(d)=\sum_{(d)}\varepsilon(d_{(1)})\cdot b\cdot f(d_{(2)})=b\cdot f(d)=f(d)\cdot b=\sum_{(d)}f(d_{(1)})\cdot\varepsilon(d_{(2)})\cdot b=(f\cdot\rho_0(b))(d).$$

☐

**Proposition 2.2.** *Let $D$ be a $k$-coalgebra and let $A$ and $B$ be $k$-algebras. If $\psi$ is an element of ${}_k\mathcal{M}(D\otimes_k A,B)$ and $\rho\in {}_k\mathcal{M}(A,{}_k\mathcal{M}(D,B))$ is the image of $\psi$ under the isomorphism (2.1), then the following are equivalent:*

1. *$\rho$ is a homomorphism of $k$-algebras and*
2. *for all $d\in D$ and all $a,b\in A$*
   - (a) *$\psi(d\otimes ab)=\sum_{(d)}\psi(d_{(1)}\otimes a)\psi(d_{(2)}\otimes b)$ and*
   - (b) *$\psi(d\otimes 1_A)=\varepsilon_D(d)1_B$.*

*Proof.* This can be seen by expanding the definition of $\rho$ and of the condition that $\rho$ be a homomorphism of $k$-algebras as is worked out in detail in [Swe69, Proposition 7.0.1]. ☐

**Definition 2.3.** *Let $D$ be a $k$-coalgebra and $A$ and $B$ be $k$-algebras. We say that $\psi\in {}_k\mathcal{M}(D\otimes_k A,B)$* measures *$A$ to $B$ if the equivalent conditions in proposition 2.2 are satisfied. The homomorphism $\psi$ is then called a $D$-*measuring *from $A$ to $B$.*

*If $A_1, A_2, B_1$ and $B_2$ are $k$-algebras, $\psi_1\colon D \otimes_k A_1 \to B_1$ measures $A_1$ to $B_1$ and $\psi_2\colon D \otimes_k A_2 \to B_2$ measures $A_2$ to $B_2$, then we say that homomorphisms $\varphi_A\colon A_1 \to A_2$ and $\varphi_B\colon B_1 \to B_2$ of $k$-algebras are* compatible with the $D$-measurings *if the diagram*

$$\begin{tikzcd}
D \otimes_k A_1 \arrow[r, "\psi_1"] \arrow[d, "\mathrm{id}_D \otimes \varphi_A"] & B_1 \arrow[d, "\varphi_B"] \\
D \otimes_k A_2 \arrow[r, "\psi_2"] & B_2
\end{tikzcd}$$

*commutes.*

*If the $k$-coalgebra $D$ contains a group-like element $1$ and $B$ is an $A$-algebra via $f\colon A \to B$, then a $D$-measuring $\psi \in {}_k\mathcal{M}(D \otimes A, B)$ is called* unital *if $\psi(1 \otimes a) = f(a)$ for all $a \in A$.*

*If $D$ is a $k$-bialgebra and $\psi \in {}_k\mathcal{M}(D \otimes_k A, A)$ is a $D$-measuring that makes $A$ into a $D$-module, then $\psi$ is called a* $D$-module algebra structure *and $(A, \psi)$ is a* $D$-module algebra. *Morphisms of $D$-module algebras are morphisms of $k$-algebras that are compatible with the $D$-measurings.*

The following lemma is clear from the definitions.

**Lemma 2.4.** *Let $D$ be a $k$-coalgebra and $A_1, A_2, B_1$ and $B_2$ be $k$-algebras. If $\psi_1 \in {}_k\mathcal{M}(D \otimes_k A_1, B_1)$ measures $A_1$ to $B_1$ and $\psi_2 \in {}_k\mathcal{M}(D \otimes_k A_2, B_2)$ measures $A_2$ to $B_2$ and $\rho_1\colon A_1 \to {}_k\mathcal{M}(D, B_1)$ and $\rho_2\colon A_2 \to {}_k\mathcal{M}(D, B_2)$ are the homomorphisms of $k$-algebras associated to $\psi_1$ and $\psi_2$, respectively, then homomorphisms of $k$-algebras $\varphi_A\colon A_1 \to A_2$ and $\varphi_B\colon B_1 \to B_2$ are compatible with the $D$-measurings if and only if the diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "\rho_1"] \arrow[d, "\varphi_A"] & {}_k\mathcal{M}(D, B_1) \arrow[d, "{}_k\mathcal{M}(D,\varphi_B)"] \\
A_2 \arrow[r, "\rho_2"] & {}_k\mathcal{M}(D, B_2)
\end{tikzcd}$$

*commutes.*

**Example 2.5.** *Let $A$ and $B$ be commutative $k$-algebras.*

(1) *Let $m$ be a natural number or $\infty$ and let $D_m := k\langle\theta^{(0)}, \dots, \theta^{(m)}\rangle$ be the free $k$-module with basis $\{\theta^{(0)}, \dots, \theta^{(m)}\}$ equipped with a $k$-coalgebra structure given by the homomorphisms of $k$-modules $\Delta\colon D_m \to D_m \otimes_k D_m$ and $\varepsilon\colon D_m \to k$ defined by*

$$\Delta(\theta^{(i)}) = \sum_{i=i_1+i_2} \theta^{(i_1)} \otimes \theta^{(i_2)} \qquad \text{and} \qquad \varepsilon(\theta^{(i)}) = \delta_{i,0} \tag{2.2}$$

*for all $i = 0, \dots, m$. A $D_m$-measuring $\psi_m\colon D_m \otimes_k A \to B$ from $A$ to $B$ is equivalent to a higher derivation $\delta = (\delta^{(0)}, \dots, \delta^{(m)})$ from $A$ to $B$ of length $m$, defined by $\delta^{(i)}(a) := \psi_m(\theta^{(i)} \otimes a)$ for all $i \in \{0, \dots, m\}$ and all $a \in A$.*

*The $k$-coalgebra $D_m$ contains the group-like element $\theta^{(0)}$. A higher derivation $\delta_A = (\delta_A^{(0)}, \dots, \delta_A^{(m)})$ of length $m$ on $A$ such that $\delta_A^{(0)}(a) = a$ for all $a \in A$ induces a $D_m$-measuring $\psi_m\colon D_m \otimes_k A \to A$ from $A$ to itself defined by $\psi_m(\theta^{(i)} \otimes a) = \delta_A^{(i)}(a)$ for all $i \in \{0, \dots, m\}$ and all $a \in A$ that is unital with respect to the group-like element $\theta^{(0)} \in D_m$.*

*The $k$-coalgebra $D_\infty$ becomes a $k$-bialgebra with respect to the $k$-algebra structure given by $1 := \theta^{(0)}$ and $\theta^{(i)} \cdot \theta^{(j)} := \binom{i+j}{i}\theta^{(i+j)}$ for all $i, j \in \mathbb{N}$. A $D_\infty$-module algebra structure on $A$ is equivalent to a unital iterative derivation $(\delta_A^{(i)})_{i\in\mathbb{N}}$ on $A$.*

(2) *We consider (1) in the special case $m = 1$. A $D_1$-measuring $\psi\colon D_1 \otimes_k A \to B$ is equivalent to a pair $(\sigma, \delta)$ consisting of homomorphism of $k$-algebras $\sigma\colon A \to B$ and a $k$-derivation $\delta\colon A \to B$, where $B$ is considered as $A$-algebra via $\sigma$ (i.e. $\delta(aa') = \delta(a)\sigma(a') + \sigma(a)\delta(a')$ for all $a, a' \in A$). This is equivalent to the data specifying a 1-differential kernel, cf. definition 1.34.*

*The $k$-coalgebra $D_1$ contains the group-like element $\theta^{(0)}$ and a $D_1$-measuring $\psi_1\colon D_1 \otimes A \to A$ from $A$ to itself that is unital with respect to $\theta^{(0)}$ in the sense that $\psi_1(\theta^{(0)} \otimes a) = a$ for all $a \in A$, is equivalent to a $k$-derivation on $A$.*

*The $k$-coalgebra $D_1$ is isomorphic to the $k$-subcoalgebra $k\langle 1, \partial\rangle$ of the $k$-bialgebra $D := k[\mathbb{G}_a] = k[\partial]$, which is the coordinate ring of the additive group scheme $\mathbb{G}_a$, with $k$-coalgebra structure given by*

$$\Delta(1) = 1 \otimes 1, \quad \varepsilon(1) = 1, \quad \Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial \qquad \text{and} \qquad \varepsilon(\partial) = 0.$$

*There is a bijection between the set of $D$-module algebra structures on $A$ and the set of $k$-derivations on $A$. The $D$-module algebra structure $\psi_A \colon D \otimes_k A \to A$ corresponding to a $k$-derivation $\delta_A \colon A \to A$ is given by $\psi_A(\partial^i \otimes a) := \delta_A^i(a)$ for all $a \in A$ and all $i \in \mathbb{N}$.*

*(3) For $n \in \mathbb{N}$ let $D_n := k\langle \sigma^0, \sigma^1 \ldots \sigma^n\rangle$ be the free $k$-module with basis $\{\sigma^0, \sigma^1, \ldots, \sigma^n\}$ with the $k$-coalgebra structure defined by*

$$\Delta(\sigma^i) := \sigma^i \otimes \sigma^i \qquad \text{and} \qquad \varepsilon(\sigma^i) = 1 \tag{2.3}$$

*for all $i = 0, \ldots, n$. We denote the element $\sigma^0 \in D_n$ also by $1$. A $D_n$-measuring $\psi_n \colon D_n \otimes_k A \to B$ from $A$ to $B$ is the same as a family of homomorphisms $\widetilde{\sigma^i} \colon A \to B$ defined by $\widetilde{\sigma^i}(a) := \psi_n(\sigma^i \otimes a)$ for $i = 0, \ldots, n$ and $a \in A$. The $k$-coalgebras $D_n$ are $k$-subcoalgebras of the $k$-bialgebra $D = k[\sigma]$ with $k$-coalgebra structure defined by (2.3) for all $i \in \mathbb{N}$. There is a bijection between the $D$-module algebra structures on $A$ and the endomorphisms of the $k$-algebra $A$.*

*(4) The coordinate ring $D = k[\mathbb{G}_m]$ of the multiplicative group scheme $\mathbb{G}_m$ over $k$ is the localization $k[\sigma, \sigma^{-1}]$ of the polynomial algebra $k[\sigma]$ over $k$ with $k$-coalgebra structure defined by equation (2.3) for all $i \in \mathbb{Z}$. There is a bijection between the set of automorphisms of the $k$-algebra $A$ and the set of $D$-module algebra structures on $A$. If $\sigma_A$ is an automorphism of the $k$-algebra $A$, then the corresponding $D$-module algebra structure $\psi_A \colon D \otimes_k A \to A$ is given by $\psi_A(\sigma^i \otimes a) := \sigma_A^i(a)$ for all $a \in A$ and $i \in \mathbb{Z}$.*

## 3. GENERALIZED DIFFERENTIALS

In this section we introduce generalized differentials and show how they specialize to objects introduced in section 1.

**Notation:** *Let $D$ be a $k$-coalgebra, $A$ be a commutative $k$-algebra, $f \colon A \to B$ be a commutative $A$-algebra, $\psi_A \colon D \otimes_k A \to A$ be a $D$-measuring from $A$ to itself, and $\rho_A \colon A \to {}_k\mathcal{M}(D, A)$ be the homomorphism of $k$-algebras associated to $\psi_A$ via the isomorphism (2.1).*

### 3.1. Definition and basic properties.

**Definition 3.1.** *(1) We define a commutative $A$-algebra*

$$\Omega^D_{B/(A,\psi_A)} := A[d(b) \mid d \in D, b \in B]/I,$$

*where $A[d(b) \mid d \in D, b \in B]$ is the polynomial algebra over $A$ in the variables $d(b)$ and $I$ is the ideal generated by the elements*

$$d(b) + d(b') - d(b + b'), \tag{3.1}$$

$$d(bb') - \sum_{(d)} d_{(1)}(b) \cdot d_{(2)}(b'), \tag{3.2}$$

$$d(f(a)) - \psi_A(d \otimes a), \tag{3.3}$$

$$(d + d')(b) - d(b) - d'(b) \text{ and} \tag{3.4}$$

$$(\lambda d)(b) - \lambda \cdot d(b) \tag{3.5}$$

*for all $d, d' \in D$, $b, b' \in B$, $\lambda \in k$ and $a \in A$. We denote the image of $d(b)$ in $\Omega^D_{B/(A,\psi_A)}$ again by $d(b)$. We also define a homomorphism of $A$-algebras*

$$\rho_u \colon B \to {}_k\mathcal{M}(D, \Omega^D_{B/(A,\psi_A)}), \quad b \mapsto (d \mapsto d(b)),$$

*where the $A$-algebra structure on ${}_k\mathcal{M}(D, \Omega^D_{B/(A,\psi_A)})$ is induced by $\rho_A \colon A \to {}_k\mathcal{M}(D, A)$ and the $A$-algebra structure on $\Omega^D_{B/(A,\psi_A)}$ that is induced by the natural $A$-algebra structure on the polynomial algebra $A[d(b) \mid d \in D, b \in B]$.*[4]

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[4] Note that $\rho_u$ is a homomorphism of $A$-algebras, since for $a \in A$ and $d \in D$ we have $\rho_u(f(a))(d) = d(f(a)) = \psi_A(d \otimes a) = \rho_A(a)(d)$.

(2) *If $D$ contains a group-like element $1$ and $\psi_A(1 \otimes a) = a$ for all $a \in A$, then $\Omega^D_{B/(A,\psi_A)}$ becomes a $B$-algebra via $b \mapsto 1(b)$, which we denote by $\Omega^{D,1}_{B/(A,\psi_A)}$.*

**Proposition 3.2.** (1) *Let $h\colon A \to R$ be a commutative $A$-algebra,*

$$\Psi\colon D \otimes_k B \to R$$

*be a $D$-measuring from $B$ to $R$, which extends the $D$-measuring $\psi_A\colon D \otimes_k A \to A$ in the sense that $\Psi(d \otimes f(a)) = h(\psi_A(d \otimes a))$ for all $d \in D$ and all $a \in A$ and let*

$$P\colon B \to {}_k\mathcal{M}(D,R)$$

*be the homomorphism of $A$-algebras associated to $\Psi$ via* (2.1)*, where the $A$-algebra structure on ${}_k\mathcal{M}(D,R)$ is given by the composition ${}_k\mathcal{M}(D,h) \circ \rho_A$. Then there exists a unique homomorphism of $A$-algebras $\phi\colon \Omega^D_{B/(A,\psi_A)} \to R$ such that the diagram*

$$\begin{tikzcd}
{}_k\mathcal{M}(D,\Omega^D_{B/(A,\psi_A)}) \arrow[rd, "{{}_k\mathcal{M}(D,\phi)}"] & \\
B \arrow[u, "\rho_u"] \arrow[r, "P"] & {}_k\mathcal{M}(D,R)
\end{tikzcd} \tag{3.6}$$

*commutes, i.e. there is a bijection*

$$\mathsf{Alg}_A(B, {}_k\mathcal{M}(D,R)) \xrightarrow{\sim} \mathsf{Alg}_A(\Omega^D_{B/(A,\psi_A)}, R), \quad P \mapsto \phi. \tag{3.7}$$

*The inverse sends $\phi \in \mathsf{Alg}_A(\Omega^D_{B/(A,\psi_A)}, R)$ to ${}_k\mathcal{M}(D,\phi) \circ \rho_u$.*

(2) *Assume that the $k$-coalgebra $D$ contains a group-like element $1$ and that $\psi_A(1 \otimes a) = a$ for all $a \in A$. Let $g\colon B \to R$ be a commutative $B$-algebra. Then for every $D$-measuring*

$$\Psi\colon D \otimes_k B \to R$$

*from $B$ to $R$, which fulfills $\Psi(1 \otimes b) = g(b)$ for all $b \in B$, which extends the $D$-measuring $\psi_A\colon D \otimes_k A \to A$ in the sense that $\Psi(d \otimes f(a)) = g(f(\psi_A(d \otimes a)))$ for all $d \in D$ and all $a \in A$, and which has*

$$P\colon B \to {}_k\mathcal{M}(D,R)$$

*as associated homomorphism of $A$-algebras, the $A$-algebra structure on ${}_k\mathcal{M}(D,R)$ being given by ${}_k\mathcal{M}(D, g \circ f) \circ \rho_A$, the unique homomorphism of $A$-algebras $\phi\colon \Omega^{D,1}_{B/(A,\psi_A)} \to R$ such that ${}_k\mathcal{M}(D,\phi) \circ \rho_u = P$ is a homomorphism of $B$-algebras. Therefore the bijection* (3.7) *restricts to a bijection*

$$\{P \in \mathsf{Alg}_A(B, {}_k\mathcal{M}(D,R)) \mid \mathrm{ev}_1 \circ P = g\} \xrightarrow{\sim} \mathsf{Alg}_B(\Omega^{D,1}_{B/(A,\psi_A)}, R), \quad P \mapsto \phi. \tag{3.8}$$

*The inverse sends $\phi \in \mathsf{Alg}_B(\Omega^{D,1}_{B/(A,\psi_A)}, R)$ to ${}_k\mathcal{M}(D,\phi) \circ \rho_u$.*

*Proof.* Let $P \in \mathsf{Alg}_A(B, {}_k\mathcal{M}(D,R))$. We first define a homomorphism of $A$-algebras

$$A[d(b) \mid d \in D, b \in B] \to R, \quad d(b) \mapsto P(b)(d). \tag{3.9}$$

This homomorphism vanishes on the ideal $I$ as defined in definition 3.1. This is clear, the only point we are explaining is the relation (3.3): Since $P\colon B \to {}_k\mathcal{M}(D,R)$ is a homomorphism of $A$-algebras and since the $A$-algebra structure on ${}_k\mathcal{M}(D,R)$ is given by ${}_k\mathcal{M}(D,h) \circ \rho_A$, we obtain $P(f(a)) = {}_k\mathcal{M}(D,h)(\rho_A(a))$ for all $a \in A$, i.e. $P(f(a))(d) = h(\rho_A(a)(d)) = h(\psi_A(d \otimes a))$. Therefore the image of $d(f(a))$ is the same as that of $\psi_A(d \otimes a)$. Hence (3.9) gives rise to a homomorphism of $A$-algebras $\phi\colon \Omega^D_{B/(A,\psi_A)} \to R$, which makes the diagram (3.6) by definition commutative. At the other side, any homomorphism of $A$-algebras $\phi\colon \Omega^D_{B/(A,\psi_A)} \to R$ is uniquely determined by its images on the elements $d(b)$ through the condition ${}_k\mathcal{M}(D,\phi) \circ \rho_u = P$. If conversely $\phi\colon \Omega^D_{B/(A,\psi_A)} \to R$ is a homomorphism of $A$-algebras, then we define $P\colon B \to {}_k\mathcal{M}(D,R)$ by $P(b)(d) := \phi(d(b))$ for all $b \in B$ and $d \in D$. This is a homomorphism of $A$-algebras, since $P(f(a))(d) = \phi(d(f(a))) = \phi(\psi_A(d \otimes a)) = h(\psi_A(d \otimes a)) = {}_k\mathcal{M}(D,h)(\rho_A(a))(d)$ for all $a \in A$ and all $d \in D$.

Now let $g\colon B \to R$ be a commutative $B$-algebra and assume that $D$ contains a group-like element $1$. If $P$ fulfills $P(b)(1) = g(b)$, then we also have $\phi(1(b)) = P(b)(1) = g(b)$ and therefore $\phi$ is the unique homomorphism of

$B$-algebras $\phi\colon \Omega^{D,1}_{B/(A,\psi)} \to R$ that fulfills ${}_k\mathcal{M}(D,\phi) \circ \rho_u = P$. Conversely, for every homomorphism of $B$-algebras $\phi\colon \Omega^{D,1}_{B/(A,\psi_A)} \to R$, the composition ${}_k\mathcal{M}(D,\phi) \circ \rho_u$ fulfills $\mathrm{ev}_1 \circ {}_k\mathcal{M}(D,\phi) \circ \rho_u(b) = \phi(1(b)) = g(b)$ for all $b \in B$. $\square$

**Remark 3.3.** *Generalized differentials can also be defined in a slightly different way: Assume that $D$ contains a group-like element $1$ and let $\psi\colon D \otimes_k A \to B$ be a unital $D$-measuring with associated homomorphism of $k$-algebras $\rho\colon A \to {}_k\mathcal{M}(D,B)$. We define $\tilde\Omega^{D,1}_{B/(A,\psi)}$ as the quotient of $A[db \mid d \in D, b \in B]$ by the ideal $I$ generated by the elements*

$$d(b) + d(b') - d(b+b'), \tag{3.10}$$

$$d(bb') - \sum_{(d)} d_{(1)}(b) \cdot d_{(2)}(b'), \tag{3.11}$$

$$d(f(a)) - 1(\psi(d \otimes a)), \tag{3.12}$$

$$(d+d')(b) - d(b) - d'(b) \text{ and} \tag{3.13}$$

$$(\lambda d)(b) - \lambda \cdot d(b) \tag{3.14}$$

*for all $b, b' \in B$, all $d, d' \in D$, all $a \in A$ and all $\lambda \in k$, and consider $\tilde\Omega^{D,1}_{B/(A,\psi)}$ as $B$-algebra via $b \mapsto 1b$. The homomorphism*

$$\rho_u\colon B \to {}_k\mathcal{M}(D, \tilde\Omega^{D,1}_{B/(A,\psi)}) \tag{3.15}$$

*will be defined as before by $\rho_u(b)(d) = db$ for all $b \in B$ and $d \in D$. If $g\colon B \to R$ is a commutative $B$-algebra and $\Psi\colon D \otimes_k B \to R$ is a $D$-measuring from $B$ to $R$, which extends the $D$-measuring $\psi\colon D \otimes_k A \to B$ in the sense that $\Psi(d \otimes f(a)) = g(\psi(d \otimes a))$ for all $d \in D$ and all $a \in A$, which is unital with respect to $1 \in D$, i.e. $\Psi(1 \otimes b) = g(b)$ for all $b \in B$, and which has*

$$P\colon B \to {}_k\mathcal{M}(D,R)$$

*as associated homomorphism of $A$-algebras (the $A$-algebra structure on ${}_k\mathcal{M}(D,R)$ is given by ${}_k\mathcal{M}(D,g) \circ \rho$), then there exists a unique homomorphism of $B$-algebras $\phi\colon \tilde\Omega^{D,1}_{B/(A,\psi)} \to R$ such that the diagram*

$$\begin{tikzcd}
{}_k\mathcal{M}(D, \tilde\Omega^{D,1}_{B/(A,\psi)}) \arrow[dr, "{{}_k\mathcal{M}(D,\phi)}"] & \\
B \arrow[u, "\rho_u"] \arrow[r, "P"'] & {}_k\mathcal{M}(D,R)
\end{tikzcd} \tag{3.16}$$

*commutes, i.e. the map*

$$\{P \in \mathsf{Alg}_A(B, {}_k\mathcal{M}(D,R)) \mid \mathrm{ev}_1 \circ P = g\} \to \mathsf{Alg}_B(\tilde\Omega^{D,1}_{B/(A,\psi)}, R), \qquad P \mapsto \phi \tag{3.17}$$

*is a bijection. The inverse sends $\phi \in \mathsf{Alg}_B(\tilde\Omega^{D,1}_{B/(A,\psi)}, R)$ to ${}_k\mathcal{M}(D,\phi) \circ \rho_u$.*

*Proof.* Given $P \in \mathsf{Alg}_A(B, {}_k\mathcal{M}(D,R))$, we first define a homomorphism of $B$-algebras

$$A[d(b) \mid d \in D, b \in B] \to R, \quad d(b) \mapsto P(b)(d),$$

where $A[d(b) \mid d \in D, b \in B]$ is considered as $B$-algebra via $b \mapsto 1(b)$. This homomorphism vanishes by assumption on the ideal $I$ as defined above. This is clear, the only point we are explaining is the relation (3.12): Since $P\colon B \to {}_k\mathcal{M}(D,R)$ is a homomorphism of $A$-algebras, we have $P(f(a)) = {}_k\mathcal{M}(D,g)(\rho(a))$ for all $a \in A$, i.e.

$$P(f(a))(d) = g(\rho(a)(d)) = g(\psi(d \otimes a)) = \Psi(1 \otimes \psi(d \otimes a)) = P(\psi(d \otimes a))(1).$$

Therefore the image of $d(f(a))$ is the same as that of $1(\psi(d \otimes a))$. Hence this homomorphism gives rise to a homomorphism of $B$-algebras $\phi\colon \tilde\Omega^{D,1}_{B/(A,\psi)} \to R$, which makes the diagram (3.16) by definition commutative. At the other side, any homomorphism of $B$-algebras $\phi\colon \tilde\Omega^{D,1}_{B/(A,\psi)} \to R$ is uniquely determined by its images on the elements $d(b)$ through the condition ${}_k\mathcal{M}(D,\phi) \circ \rho_u = P$. $\square$

3.2. **$\times_A$-bialgebras and prolongations.** We briefly recall the definition of $\times_A$-bialgebras, which were introduced by Sweedler (cf. [Swe74]), following closely the exposition of Masuoka and Yanagawa in [MY13].

**Assumption 3.4.** *We assume that $k$ is a field and let $\mathcal{A}$ be an $A$-ring that is projective as left $A$-module.*

Its $A$-ring structure makes $\mathcal{A}$ into an $A$-$A$-bimodule. We consider the tensor product of the left $A$-module $\mathcal{A}$ with itself over $A$ (ignoring its right $A$-module structure, but denoting it by abuse of notation by $\mathcal{A} \otimes_A \mathcal{A}$). Its subset

$$\mathcal{A} \times_A \mathcal{A} := \Big\{ \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A} \otimes_A \mathcal{A} \mid \sum_{i=1}^{n} a_i x \otimes b_i = \sum_{i=1}^{n} a_i \otimes b_i x \; \forall x \in A \Big\}.$$

becomes an $A$-ring with respect to $a \mapsto a \otimes 1$ for all $a \in A$.[5] We assume in addition that $\mathcal{A}$ is an $A$-coalgebra with respect to $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes_A \mathcal{A}$ and $\varepsilon \colon \mathcal{A} \to A$.

**Definition 3.5.** *We call $\mathcal{A}$ a $\times_A$*-bialgebra *if the following hold:*

*(1)* $\Delta(\mathcal{A}) \subseteq \mathcal{A} \times_A \mathcal{A}$,
*(2)* $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes_A \mathcal{A}$ *is a morphism of $A$-ring,*
*(3)* $\varepsilon(1) = 1$ *and*
*(4)* $\varepsilon(ab) = \varepsilon(a\varepsilon(b))$ *for all $a, b \in \mathcal{A}$.*

If $M$ and $N$ are left $\mathcal{A}$-modules (which we also consider as symmetric $A$-bimodules), then the tensor product $M \otimes_A N$ becomes a left $\mathcal{A}$-module with respect to the action given by

$$\alpha \rightharpoonup (m \otimes n) = \sum_{(\alpha)} (\alpha_{(1)} \rightharpoonup m) \otimes (\alpha_{(2)} \rightharpoonup n) \tag{3.18}$$

for all $\alpha \in \mathcal{A}$, $m \in M$ and $n \in N$. This is well defined because of condition (1) above and this action is associative by condition (2). From condition (3) and (4) we obtain that $A$ is a left $\mathcal{A}$-module with respect to the action given by

$$\alpha \rightharpoonup a := \varepsilon(\alpha a) \tag{3.19}$$

for all $\alpha \in \mathcal{A}$ and $a \in A$.

We recall the following results of Masuoka and Yanagawa.

**Proposition 3.6** ([MY13, Proposition 2.2])**.** *Let $\mathcal{A}$ be a $\times_A$-bialgebra. Then the category of left $\mathcal{A}$-modules is a monoidal category with respect to the product given by $M \otimes_A N$ for any two left $\mathcal{A}$-modules $M$ and $N$, considered as left $\mathcal{A}$-module via* (3.18)*, and the unit given by $A$, with the left $\mathcal{A}$-module structure given by* (3.19)*. If $\mathcal{A}$ is cocommutative as $A$-coalgebra, then this monoidal category is symmetric with symmetry given by the interchange of factors.*

If $D$ is a $k$-bialgebra and $A$ is a commutative $D$-module algebra, then the smash product $A\#D$ (cf. [Swe69, p. 153]) is an $A$-ring via $A \to A\#D, a \mapsto a\#1$ and an $A$-coalgebra via the base extension of $D$ from $k$ to $A$.

**Lemma 3.7** ([MY13, Lemma 2.3])**.** *If $D$ is a cocommutative $k$-bialgebra and $A$ is a commutative $D$-module algebra, then the smash product $A\#D$ is a $\times_A$-bialgebra.*

Under the hypothesis in lemma 3.7, the smash product $A\#D$ is a $\times_A$-bialgebra and by proposition 3.6 the category of left $(A\#D)$-modules is a symmetric monoidal category with respect to the tensor product over $A$ considered as left $\mathcal{A}$-module via (3.18) and the unit (3.19) with symmetry given by the interchange of factors. We refer to this symmetric monoidal category as the *category of left $(A\#D)$-modules over $A$.*

**Remark 3.8.** *(1) Let $m$ be a natural number or $\infty$ and $D_m$ be the $k$-coalgebra defined in example 2.5 (1). By this example, a higher derivation $\delta_A = (\delta_A^{(i)})_{i=0,\dots,m}$ of length $m$ gives rise to a $D$-measuring $\psi_m \colon D_m \otimes_k A \to A$ from $A$ to itself. Given a commutative higher differential ring $(A, \delta_A)$, a* higher differential module *over $(A, \delta_A)$ is an $A$-module $M$ together with a family of additive maps $(\partial^{(i)} \colon M \to M)_{i=0,\dots,m}$ such that $\partial^{(i)}(am) = \sum_{i=i_1+i_2} \delta_A^{(i_1)}(a)\partial^{(i_2)}(m)$ for all $a \in A$ and $m \in M$.*[6] *A morphism from a higher differential*

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[5] A more systematic introduction of $\mathcal{A} \times_A \mathcal{A}$ is given in [Swe74].

[6] Some authors require in addition that $\partial^{(0)} = \mathrm{id}_M$.

module $(M, \partial_M)$ to $(N, \partial_N)$ over $(A, \delta_A)$ is a homomorphism of $A$-modules $f\colon M \to N$ such that $\partial_N^{(i)} \circ f = f \circ \partial_M^{(i)}$ for all $i = 0, \dots, m$. A higher differential module $(M, (\partial^{(i)})_{i=0,\dots,m})$ is unital if $\partial^{(0)} = \mathrm{id}_M$. If $m = \infty$, then a higher differential module $(M, (\partial^{(i)})_{i\in\mathbb{N}})$ is iterative if $\partial^{(i)} \circ \partial^{(j)} = \binom{i+j}{i}\partial^{(i+j)}$ for all $i, j \in \mathbb{N}$. The category of higher differential modules over $(A, \delta_A)$ is monoidal, the product of $(M, \partial_M)$ and $(N, \partial_N)$ being the $A$-module $M \otimes_A N$ with higher derivation $\partial = (\partial^{(i)})_{i=0,\dots,m}$ defined by $\partial^{(i)}(m \otimes n) := \sum_{i=i_1+i_2} \partial_M^{(i_1)}(m) \otimes \partial_N^{(i_2)}(n)$ for all $m \otimes n \in M \otimes_A N$ and all $i \in \{0, \dots, m\}$ and unit $(A, \delta_A)$. It is moreover symmetric with respect to the interchange of factors. The unital (and/or iterative, in the case $m = \infty$) higher differential modules over $(A, \delta_A)$ form a full subcategory that is again symmetric monoidal. If $m = \infty$, then the symmetric monoidal category of left $(A\#D_\infty)$-modules over $A$ is isomorphic to the category of unital iterative differential modules over $(A, \delta_A)$. Commutative monoids in the symmetric monoidal category of left $(A\#D_\infty)$-modules over $A$ are commutative unital iterative differential $(A, \delta_A)$-algebras.

(2) Let $D = k[\mathbb{G}_a]$ be the coordinate ring of the additive group scheme $\mathbb{G}_a$ over $k$. Let $(A, \delta_A)$ be a commutative differential $k$-algebra. By example 2.5 (2) a $D$-module algebra structure $\psi_A\colon D \otimes_k A \to A$ is associated to $\delta_A$. The smash product $A\#D$ is isomorphic to the ring of differential operators $A[\partial]$, which is the ring consisting of elements of the form $a_m\partial^m + \dots + a_1\partial + a_0$ with $a_i \in A$ and commutation relation defined by $\partial a = a\partial + \delta_A(a)$ for all $a \in A$. If $A$ contains an element $a$ such that $\delta_A(a) \neq 0$, then the ring $A[\partial]$ is not commutative. A differential module over $(A, \delta_A)$ is an $A$-module together with an additive map $\partial_M : M \to M$ such that $\partial_M(am) = \delta_A(a)m + a\partial_M(m)$ for all $a \in A$ and $m \in M$. A morphism from a differential module $(M, \partial_M)$ to $(N, \partial_N)$ is a homomorphism of $A$-modules $f\colon M \to N$ such that $\partial_N \circ f = f \circ \partial_M$. The category of differential modules over $(A, \delta_A)$ is monoidal, the product of $(M, \partial_M)$ and $(N, \partial_N)$ being given by the tensor product $M \otimes_A N$ with derivation $\partial$ defined by $\partial(m \otimes n) := \partial_M(m) \otimes n + m \otimes \partial_N(n)$ and the unit being $(A, \delta_A)$, cf. [vdPS03, Chapter 2]. It is furthermore symmetric with respect to the interchange of factors. The symmetric monoidal category of differential modules over $(A, \delta_A)$ is isomorphic to the symmetric monoidal category of left $A\#D$-modules over $A$ as defined above. Commutative monoids in the category of $A[\partial]$-modules over $A$ are commutative differential $(A, \delta_A)$-algebras, cf. definition 1.23.

(3) Let $D$ be the coordinate ring $k[\mathbb{G}_m] \cong k[\sigma, \sigma^{-1}]$ of the multiplicative group scheme $\mathbb{G}_m$ over $k$. As noted in example 2.5 (4), there is a bijection between the set of automorphisms of the $k$-algebra $A$ and the set of $D$-module algebra structures on $A$. Let $\sigma_A$ be an automorphism of the $k$-algebra $A$ and $\psi_A\colon D \otimes_k A \to A$ be the corresponding $D$-module algebra structure on $A$. The category of inversive difference modules over $(A, \sigma_A)$ consists of $A$-modules $M$ together with an automorphism $\Sigma$ of the abelian group $M$ such that $\Sigma(am) = \sigma_A(a)\Sigma(m)$ for all $a \in A$ and $m \in M$. A morphism of inversive difference modules from $(M, \Sigma_M)$ to $(N, \Sigma_N)$ is a morphism of $A$-modules $f\colon M \to N$ such that $\Sigma_N \circ f = f \circ \Sigma_M$. The category of inversive difference modules over $(A, \sigma_A)$ is monoidal, the product of two inversive difference modules $(M, \Sigma_M)$ and $(N, \Sigma_N)$ being the $A$-module $M \otimes_A N$ with the automorphism $\Sigma$ defined by $\Sigma(m \otimes n) := \Sigma_M(m) \otimes \Sigma_N(n)$ for all $m \otimes n \in M \otimes_A N$ and the unit being $(A, \sigma_A)$. It is symmetric with respect to the interchange of factors. The symmetric monoidal category of inversive difference modules over $(A, \sigma_A)$ is isomorphic to the category of left $(A\#D)$-modules over $A$.

(4) Let $D$ be the $k$-subbialgebra $k[\sigma]$ of $k[\mathbb{G}_m] \cong k[\sigma, \sigma^{-1}]$. Then commutative $D$-module algebras are in bijection with commutative $k$-algebras $A$ equipped with an endomorphism $\sigma_A$, cf. example 2.5 (3). Let $(A, \sigma_A)$ be a commutative difference algebra over $k$ and $\psi_A\colon D \otimes_k A \to A$ be the associated $D$-module algebra structure on $A$. The category of difference modules over $(A, \sigma_A)$ consists of $A$-modules $M$ equipped with an endomorphism $\Sigma$ of the abelian group $M$ such that $\Sigma(am) = \sigma_A(a)\Sigma(m)$ for all $a \in A$ and $m \in M$ and morphisms defined as in the case of inversive difference modules. This category is a symmetric monoidal category in a similar way as the category of inversive difference modules. It is isomorphic to the symmetric monoidal category of left $(A\#D)$-modules over $A$.

**Proposition 3.9.** Let $I$ be a directed set and $\gamma\colon I \times I \to I$ be a map such that for all $i, j \in I$ we have $i \leq \gamma(i, j)$ and $j \leq \gamma(i, j)$. Let $k$ be a field and let $D$ be a cocommutative $k$-bialgebra that is the direct limit of a directed system of $k$-subcoalgebras $(D_i)_{i\in I}$ such that $D_iD_j \subseteq D_{\gamma(i,j)}$ for all $i, j \in I$ and such that every $D_i$ contains the unit $1$ of $D$. Let further $\psi_A\colon D \otimes_k A \to A$ be a $D$-module algebra structure on $A$ and denote the $D_i$-measurings from $A$ to itself induced by $\psi_A$ by $\psi_{A,i}\colon D_i \otimes_k A \to A$.

(1) *Then the $B$-algebras $\left(\Omega^{D_i,1}_{B/(A,\psi_{A,i})}\right)_{i\in I}$ form a direct system and there is an isomorphism of $B$-algebras*

$$\Omega^{D,1}_{B/(A,\psi_A)} \cong \varinjlim_{i\in I} \Omega^{D_i,1}_{B/(A,\psi_{A,i})}. \tag{3.20}$$

*Furthermore, for all $i,j \in I$ the $k$-coalgebra $D_i$ measures $\Omega^{D_j,1}_{B/(A,\psi_{A,j})}$ to $\Omega^{D_{\gamma(i,j)},1}_{B/(A,\psi_{A,\gamma(i,j)})}$ and $\Omega^{D,1}_{B/(A,\psi_A)}$ becomes a $D$-module algebra. Moreover, $\Omega^{D,1}_{B/(A,\psi_A)}$ is a commutative monoid in the symmetric monoidal category of left $(A\#D)$-modules over $A$ with respect to the $(A\#D)$-module structure*

$$\Psi\colon (A\#D)\otimes \Omega^{D,1}_{B/(A,\psi_A)} \to \Omega^{D,1}_{B/(A,\psi_A)}.$$

(2) *For every commutative monoid $(S,\psi_S\colon (A\#D)\otimes S \to S)$ in the category of left $(A\#D)$-modules over $A$ and every homomorphism of $A$-algebras $g\colon B\to S$ there exists a unique homomorphism of monoids of left $(A\#D)$-modules over $A$*

$$G\colon (\Omega^{D,1}_{B/(A,\psi_A)},\Psi) \to (S,\psi_S)$$

*such that the diagram*

$$\begin{tikzcd} B \arrow[r, "\rho_0"] \arrow[dr, "g"'] & \Omega^{D,1}_{B/(A,\psi_A)} \arrow[d, dotted, "G"] \\ & S \end{tikzcd}$$

*commutes, where $\rho_0$ is the $B$-algebra structure of $\Omega^{D,1}_{B/(A,\psi_A)}$. This induces a bijection between the set of homomorphisms of monoids of left $(A\#D)$-modules over $A$ from $(\Omega^{D,1}_{B/(A,\psi_A)},\Psi)$ to $(S,\psi_S)$ and the set of homomorphisms of $A$-algebras from $B$ to $S$.*

*The commutative monoid $(\Omega^{D,1}_{B/(A,\psi_A)},\Psi)$ in the category of left $(A\#D)$-modules over $A$ is unique with this property.*

*Proof.* For $i\in\mathbb{N}$ we denote by $I_i$ the ideal of $A[db \mid d\in D_i, b\in B]$ that is generated by the elements

$$\begin{aligned} d(b)+d(b')-d(b+b'),\\ d(bb')-\sum_{(d)} d_{(1)}(b)\cdot d_{(2)}(b'),\\ d(f(a))-\psi_{A,i}(d\otimes a),\\ (d+d')(b)-d(b)-d'(b) \text{ and}\\ (\lambda d)(b)-\lambda\cdot d(b) \end{aligned}$$

for all $d,d'\in D_i$, $b,b'\in B$, $\lambda\in k$ and $a\in A$. Then $\Omega^{D_i,1}_{B/(A,\psi_{A,i})} \cong A[db \mid d\in D_i, b\in B]/I_i$. If $i\le j$ are elements of $I$, then there is an injection from $A[db\mid d\in D_i, b\in B]$ into $A[db \mid d\in D_j, b\in B]$ and the image of $I_i$ under this injection is contained in $I_j$. Therefore we obtain an injection $\Omega^{D_i,1}_{B/(A,\psi_{A,i})} \to \Omega^{D_j,1}_{B/(A,\psi_{A,j})}$ and the isomorphism (3.20). We consider the morphism of $k$-modules

$$\begin{aligned} D_i\otimes_k A[db \mid d\in D_j, b\in B] &\to \Omega^{D_{\gamma(i,j)},1}_{B/(A,\psi_{A,\gamma(i,j)})}\\ d\otimes a(d'_2b_2)\dots(d'_nb_n) &\mapsto \sum_{(d)} d_{(1)}(f(a))\cdot(d_{(2)}d'_2)(b_2)\dots(d_{(n)}d'_n)(b_n), \end{aligned}$$

using Sweedler notation, for all $a\in A$, $b_2,\dots,b_n\in B$, $d\in D_i$ and $d'_2,\dots,d'_n\in D_j$. Since the image of $D_i\otimes_k I_j$ is zero in $\Omega^{D_{\gamma(i,j)},1}_{B/(A,\psi_{A,\gamma(i,j)})}$, we obtain a morphism of $k$-modules

$$\tilde{\Psi}_{i,j}\colon D_i\otimes_k \Omega^{D_j,1}_{B/(A,\psi_{A,j})} \to \Omega^{D_{\gamma(i,j)},1}_{B/(A,\psi_{A,\gamma(i,j)})},$$

which is by construction a $D_i$-measuring from $\Omega^{D_j,1}_{B/(A,\psi_{A,j})}$ to $\Omega^{D_{\gamma(i,j)},1}_{B/(A,\psi_{A,\gamma(i,j)})}$. By the universal property of the direct limit

$$D \otimes_k \Omega^{D,1}_{B/(A,\psi_A)} \cong \varinjlim_{i\in I} D_i \otimes_k \varinjlim_{j\in I} \Omega^{D_j,1}_{B/(A,\psi_{A,j})} \cong \varinjlim_{i,j\in I\times I} \left(D_i \otimes_k \Omega^{D_j,1}_{B/(A,\psi_{A,j})}\right)$$

we obtain a morphism $\tilde{\Psi}\colon D \otimes_k \Omega^{D,1}_{B/(A,\psi_A)} \to \Omega^{D,1}_{B/(A,\psi_A)}$ such that the following diagram commutes

$$\begin{tikzcd}
D_i \otimes_k \Omega^{D_j,1}_{B/(A,\psi_{A,j})} \arrow[r, "\tilde{\Psi}_{i,j}"] \arrow[d, hook] & \Omega^{D_{\gamma(i,j)},1}_{B/(A,\psi_{A,\gamma(i,j)})} \arrow[d, hook] \\
D \otimes_k \Omega^{D,1}_{B/(A,\psi_A)} \arrow[r, "\tilde{\Psi}"] & \Omega^{D,1}_{B/(A,\psi_A)}.
\end{tikzcd}$$

The morphism $\tilde{\Psi}$ is a $D$-measuring from $\Omega^{D,1}_{B/(A,\psi_A)}$ to itself, since the $\tilde{\Psi}_{i,j}$ are $D_i$-measurings. By definition $\tilde{\Psi}$ makes $\Omega^{D,1}_{B/(A,\psi_A)}$ also into a left $D$-module. We extend this to a left $(A\#D)$-module structure

$$\Psi\colon (A\#D) \otimes \Omega^{D,1}_{B/(A,\psi_A)} \to \Omega^{D,1}_{B/(A,\psi_A)}, \quad (a\#d)\otimes\omega \mapsto a \cdot \tilde{\Psi}(d\otimes\omega).$$

Let $S$ be a commutative monoid in the symmetric monoidal category of left $(A\#D)$-modules over $A$ with $(A\#D)$-module structure given by $\psi_S\colon (A\#D)\otimes S \to S$. We denote the $A$-algebra structure on $S$ by $h\colon A \to S$. Let $g\colon B \to S$ be a homomorphism of $A$-algebras. We define

$$G\colon \Omega^{D,1}_{B/(A,\psi_A)} \to S$$

by

$$G(a_1 d_2(b_2)\dots d_n(b_n)) \coloneqq h(a_1)\cdot \psi_S((1\#d_2)\otimes g(b_2)) \cdot \ldots \cdot \psi_S((1\#d_n)\otimes g(b_n))$$

for all $a_1 \in A$, all $b_2, \dots b_n \in B$ and all $d_2, \dots, d_n \in D$. If furthermore $a \in A$ and $d \in D$, then we have

$$\begin{aligned}
&G\left(\Psi((a\#d)\otimes a_1 d_2'(b_2)\dots d_n'(b_n))\right)\\
&= G\Big(a \sum_{(d)} d_{(1)}(f(a_1))(d_{(2)}d_2')(b_2)\dots(d_{(n)}d_n')(b_n)\Big)\\
&= h(a)\sum_{(d)} \psi_S((1\#d_{(1)})\otimes g(f(a_1)))\cdot\psi_S((1\#d_{(2)}d_2')\otimes g(b_2))\dots\psi_S((1\#d_{(n)}d_n')\otimes g(b_n))\\
&= h(a)\sum_{(d)} \psi_S((1\#d_{(1)})\otimes h(a_1))\cdot\psi_S(1\#d_{(2)}\otimes\psi_S((1\#d_2')\otimes g(b_2)))\dots\psi_S((1\#d_{(n)})\otimes\psi_S((1\#d_n')\otimes g(b_n)))\\
&= h(a)\cdot\psi_S((1\#d)\otimes h(a_1)\cdot\psi_S((1\#d_2')\otimes g(b_2))\dots\psi_S((1\#d_n')\otimes g(b_n)))\\
&= \psi_S((a\#d)\otimes G(a_1 d_2'(b_2)\dots d_n'(b_n))),
\end{aligned}$$

so that $G$ is a morphism of left $(A\#D)$-modules. By definition $G$ is multiplicative and respects the units. Therefore $G$ is a morphism of monoids of left $(A\#D)$-modules over $A$. In order to show that $G\colon \Omega^{D,1}_{B/(A,\psi_A)} \to S$ is unique, let $G'\colon \Omega^{D,1}_{B/(A,\psi_A)} \to S$ be another morphism of monoids of left $(A\#D)$-modules over $A$ such that $G'\circ\rho_0 = g$. Then we have $G'(a\cdot d(b)) = G'(\Psi((a\#d)\otimes 1b)) = \psi_S((a\#d)\otimes G'(1b)) = \psi_S((a\#d)\otimes g(b)) = \psi_S((a\#d)\otimes G(1b)) = G(\Psi((a\#d)\otimes 1b)) = G(a\cdot d(b))$ for all $a\in A$, all $b \in B$ and all $d\in D$, and therefore $G' = G$. If conversely $G\colon \Omega^{D,1}_{B/(A,\psi_A)} \to S$ is a morphism of monoids of left $(A\#D)$-modules over $A$, then $G\circ\rho_0$ is a morphism of $A$-algebras from $B$ to $S$.

The uniqueness of $(\Omega^{D,1}_{B/(A,\psi_A)}, \Psi)$ follows from its universal property. ☐

**Remark 3.10.** *(1) If $D$ is a $k$-bialgebra, then the finitely generated $k$-subcoalgebras of $D$ form a directed set and the conditions at the beginning of the last proposition can always be satisfied.*

*(2) Let $I$ be the set $\mathbb{N}$ of natural numbers with the natural partial order.*

  *(a) The free $k$-module $D_i \coloneqq k\langle\partial^0,\dots,\partial^i\rangle$ is a $k$-subcoalgebra of $D = k[\mathbb{G}_a] = k[\partial]$ for all $i\in\mathbb{N}$, cf. example 2.5 (2). We recover proposition 1.24 from proposition 3.9, cf. example 3.12 for details.*

*(b) The free $k$-module $D_i := k\langle\sigma^0, \dots, \sigma^i\rangle$ is a $k$-subcoalgebra of $D = k[\sigma]$ for all $i \in \mathbb{N}$, where $\sigma^j$ are group-like elements for all $j \in \mathbb{N}$. We recover proposition 1.32 from proposition 3.9, cf. example 3.14 for details.*

*(c) The $k$-bialgebra $k\langle\theta^{(i)} \mid i \in \mathbb{N}\rangle$ defined in example 2.5 (1) is the direct limit of its $k$-subcoalgebras $(k\langle\theta^{(0)}, \dots, \theta^{(i)}\rangle)_{i\in\mathbb{N}}$. We recover proposition 1.22 from proposition 3.9, cf. example 3.11 for details.*

*(d) More generally, any iterative Hasse-Schmidt system $\mathcal{D} = (\mathcal{D}_i)_{i\in\mathbb{N}}$ in the sense of Moosa and Scanlon, cf.* [MS11, Definition 2.2], *gives rise to a direct system of cocommutative $k$-coalgebras $(\mathcal{D}_i(k)^*)_{i\in\mathbb{N}}$ such that $D := \varinjlim_{i\in\mathbb{N}} \mathcal{D}_i(k)^*$ is a $k$-bialgebra, cf.* [Hei13b].

## 3.3. **Examples.**

### 3.3.1. *Derivations and higher derivations.*

**Example 3.11.** *Let $m$ be a natural number and let $D_m := k\langle\theta^{(0)}, \dots, \theta^{(m)}\rangle$ be the $k$-coalgebra defined in example 2.5 (1). If $\psi_m \colon D_m \otimes_k A \to A$ is a $D_m$-measuring from $A$ to itself, then we obtain*

$$\Omega^{D_m}_{B/(A,\psi_m)} = A[\theta^{(0)}(b), \dots, \theta^{(m)}(b) \mid b \in B]/I,$$

*where $I$ is the ideal generated by*

$$\theta^{(i)}(b + b') - \theta^{(i)}(b) - \theta^{(i)}(b'), \quad \theta^{(i)}(bb') - \sum_{i_1+i_2=i} \theta^{(i_1)}(b)\theta^{(i_2)}(b') \quad \text{and} \quad \theta^{(i)}(f(a)) - \psi_m(\theta^{(i)} \otimes a)$$

*for all $b, b' \in B$, all $a \in A$ and all $i = 0, \dots, m$. A $D_m$-measuring $\psi_m \colon D_m \otimes_k A \to A$ from $A$ to itself is equivalent to a higher derivation $\delta_A = (\delta_A^{(0)}, \dots, \delta_A^{(m)})$ on $A$ of length $m$, defined by $\delta_A^{(i)}(a) := \psi_m(\theta^{(i)} \otimes a)$ for all $i \in \{0, \dots, m\}$ and all $a \in A$.*

*The $k$-coalgebra $D_m$ contains the group-like element $\theta^{(0)}$. A higher derivation $\delta_A = (\delta_A^{(0)}, \dots, \delta_A^{(m)})$ of length $m$ on $A$ such that $\delta_A^{(0)}(a) = a$ for all $a \in A$ induces a $D_m$-measuring $\psi_m \colon D_m \otimes_k A \to A$ from $A$ to itself defined by $\psi_m(\theta^{(i)} \otimes a) = \delta_A^{(i)}(a)$ for all $i \in \{0, \dots, m\}$ and all $a \in A$, which is unital with respect to the group-like element $\theta^{(0)} \in D_m$. The $B$-algebra structure on $\Omega^{D_m,\theta^{(0)}}_{B/(A,\psi_m)}$ is given by $b \mapsto \theta^{(0)}(b)$ and $\Omega^{D_m,\theta^{(0)}}_{B/(A,\psi_m)}$ is isomorphic to the $B$-algebra $\mathrm{HS}^m_{B/(A,\delta_A)}$ defined by Rosen, cf. definition 1.15. The bijection* (3.7) *specializes in this case to* (1.12).

*If the $D_m$-measuring $\psi_m \colon D_m \otimes_k A \to A$ is trivial, i.e. $\psi_m(\theta^{(i)} \otimes a) = a \cdot \delta_{i,0}$ for all $i \in \{0, \dots, m\}$, then $\Omega^{D_m,\theta^{(0)}}_{B/(A,\psi_m)}$ specializes to the $B$-algebra $\mathrm{HS}^m_{B/A}$ defined by Vojta, cf. definition 1.7, and the bijection* (3.7) *specializes to* (1.6) *(in the case where $m$ is a natural number, the case $m = \infty$ is similar).*

*Let $\delta_A = (\delta_A^{(i)})_{i\in\mathbb{N}}$ be a unital iterative higher derivation on $A$, $D_\infty := k\langle\theta^{(i)} \mid i \in \mathbb{N}\rangle$ and $\psi_A \colon D_\infty \otimes_k A \to A$ be the corresponding $D_\infty$-module algebra structure on $A$, cf. example 2.5 (1). Then the $D_\infty$-module algebra $(\Omega^{D_\infty,\theta^{(0)}}_{B/(A,\psi_A)}, \Psi)$ is isomorphic to the unital iterative differential ring $(\mathrm{HS}^\infty_{B/(A,\delta_A)}, d)$ defined by Rosen, cf. definition 1.18 and lemma 1.21. By remark 3.8 (1) commutative monoids in the symmetric monoidal category of $(A\#D_\infty)$-modules over $A$ are commutative unital iterative higher differential $(A, \delta_A)$-algebras and we recognize proposition 1.22 as a corollary of proposition 3.9.*

**Example 3.12.** *If $m = 1$ in example 3.11, then*

$$D_1 = k\langle\theta^{(0)}, \theta^{(1)}\rangle =: k\langle 1, \partial\rangle.$$

*Let $\psi_A \colon D_1 \otimes_k A \to A$ be a $D_1$-measuring from $A$ to itself, which is equivalent to a pair $(\sigma_A, \delta_A)$ consisting of an endomorphism $\sigma_A$ of the $k$-algebra $A$ and a $k$-derivation $\delta_A \colon A \to A$, where $A$ is considered as $A$-algebra via $\sigma_A$ (i.e. $\delta_A(aa') = \delta_A(a)\sigma_A(a') + \sigma_A(a)\delta_A(a')$). The homomorphism of $k$-algebras $\rho_A$ associated to $\psi_A$ is then given by the homomorphism $A \to A[t]/(t^2)$ that sends $a \in A$ to $\sigma_A(a) + \delta_A(a)t$. We have*

$$\Omega^{D_1}_{B/(A,\psi_A)} = A[1(b), \partial(b) \mid b \in B]/I,$$

*where $I$ is the ideal generated by*

$$\begin{aligned} &1(b+b')-1(b)-1(b'), &&\partial(b+b')-\partial(b)-\partial(b'),\\ &1(bb')-1(b)1(b'), &&\partial(bb')-1(b)\partial(b')-\partial(b)1(b'),\\ &1(f(a))-\psi_A(1\otimes a), &&\partial(f(a))-\psi_A(\partial\otimes a)\end{aligned}$$

*for all $a\in A$ and $b\in B$.*

*Let $h\colon A\to R$ be a commutative $A$-algebra. Then ${}_k\mathcal{M}(D_1,R)$ is isomorphic to $R[t]/(t^2)$, which we consider as $A$-algebra via the homomorphism ${}_k\mathcal{M}(D_1,h)\circ\rho_A$, which sends $a\in A$ to $h(\sigma_A(a))+h(\delta_A(a))t$. In this case the bijection (3.7) takes the form*

$$\mathsf{Alg}_A(B,R[t]/(t^2))\xrightarrow{\sim}\mathsf{Alg}_A(\Omega^{D_1}_{B/(A,\psi_A)},R),\quad P\mapsto\phi. \tag{3.21}$$

*The data specified by an element $\phi\in\mathsf{Alg}_A(\Omega^{D_1}_{B/(A,\psi_A)},R)$, or in view of (3.21) equivalently by an element $P\in\mathsf{Alg}_A(B,R[t]/(t^2))$, is given by a homomorphism of $k$-algebras $\Sigma\colon B\to R$ and a $k$-derivation $\partial\colon B\to R$, where $R$ is considered as $B$-algebra via $\Sigma\colon B\to R$ (i.e. $\partial(bb')=\partial(b)\Sigma(b')+\Sigma(b)\partial(b')$), such that the diagrams*

$$\begin{tikzcd} A \arrow[r, "\sigma_A"] \arrow[d, "f"] & A \arrow[d, "h"] \\ B \arrow[r, "\Sigma"] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\delta_A"] \arrow[d, "f"] & A \arrow[d, "h"] \\ B \arrow[r, "\partial"] & R \end{tikzcd}$$

*commute.*

*Let $A\xrightarrow{(f,\delta)}B$ be a 1-differential kernel in the sense of Johnson (cf. subsection 1.8). We define the $D_1$-measuring $\psi\colon D_1\otimes_k A\to B$ by $\psi(1\otimes a):=f(a)$ and $\psi(\partial\otimes a):=\delta(a)$ and let $\rho\colon A\to B[t]/(t^2), a\mapsto f(a)+\delta(a)t$ be the associated homomorphism of $k$-algebras. The homomorphism of $A$-algebras $\rho_u\colon B\to\tilde\Omega^{D_1,1}_{B/(A,\psi)}$ (cf. (3.15)) provides a prolongation of $(f,\delta)$ consisting of the ring homomorphism*

$$g\colon B\to\tilde\Omega^{D_1,1}_{B/(A,\psi)},\quad b\mapsto 1b\ (=\rho_u(b)(1))$$

*and the derivation*

$$\partial\colon B\to\tilde\Omega^{D_1,1}_{B/(A,\psi)},\quad b\mapsto\partial b\ (=\rho_u(b)(\partial)).$$

*This prolongation is universal in the sense that for any prolongation $B\xrightarrow{(g',\partial')}C'$ there is a homomorphism of $B$-algebras $h\colon\Omega^{D_1,1}_{B/(A,\psi)}\to C'$ such that $g'=h\circ g$ and $\partial'=h\circ\partial$. In fact, if $(g',\partial')$ is such a prolongation, then we have $g'\circ\delta=\partial'\circ f$ and therefore*

$$P'\colon B\to{}_k\mathcal{M}(D_1,C')\ (\cong C'[t]/(t^2)),\qquad b\mapsto g'(b)+\partial'(b)t$$

*is a homomorphism of $A$-algebras, when we consider ${}_k\mathcal{M}(D_1,C')\cong C'[t]/(t^2)$ as $A$-algebra via ${}_k\mathcal{M}(D_1,g')\circ\rho$, and satisfies $\varepsilon\circ P'=g'$, where $\varepsilon\colon C'[t]/(t^2)\to C$ is the homomorphism of $C$-algebras defined by $\varepsilon(t):=0$. By remark 3.3 there exists a unique homomorphism of $B$-algebras $\phi\colon\tilde\Omega^{D_1,1}_{B/(A,\psi)}\to C'$ such that $P'={}_k\mathcal{M}(D_1,\phi)\circ\rho_u$. This means that $g'=\phi\circ f$ and $\partial'=\phi\circ\delta$. Therefore proposition 1.36 becomes a corollary of remark 3.3.*

*The $k$-coalgebra $D_1$ contains the group-like element $1=\theta^{(0)}$. If $\psi_A(1\otimes a)=a$ for all $a\in A$, then the $B$-algebra $\Omega^{D_1,1}_{B/(A,\psi_A)}$ is isomorphic to the quotient of $B[\partial(b)\mid b\in B]$ by the ideal generated by*

$$\partial(b+b')-\partial(b)-\partial(b'),\quad \partial(bb')-b\partial(b')-b'\partial(b)\quad\text{and}\quad\partial(f(a))-f(\psi_A(\partial\otimes a))$$

*for all $b,b'\in B$ and all $a\in A$, which is isomorphic to the $B$-algebra $\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A}$ as defined in subsection 1.2, cf. remark 1.3.*

*Now assume that $R$ is a commutative $B$-algebra via $g\colon B\to R$. Then the bijection (3.8) takes the special form*

$$\{P\in\mathsf{Alg}_A(B,R[t]/(t^2))\mid\varepsilon\circ P=g\}\xrightarrow{\sim}\mathsf{Alg}_B(\mathrm{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{f\circ\delta_A},R),\quad P\mapsto\phi, \tag{3.22}$$

*where $\varepsilon\colon R[t]/(t^2)\to R$ is the homomorphism of $R$-algebras defined by $\varepsilon(t)=0$. An element $P\colon B\to R[t]/(t^2)$ of the set on the left hand side of (3.22) is given by $P(b)=g(b)+\partial(b)t$ with $\partial\in\mathrm{Der}_{\delta_A}(B,R)$. We recover lemma 1.2.*

*If the $D_1$-measuring $\psi_A\colon D_1 \otimes_k A \to A$ is trivial (i.e. $\psi_A(1 \otimes a) = a$ and $\psi_A(\partial \otimes a) = 0$ for all $a \in A$), then the $B$-algebra $\Omega^{D_1,1}_{B/(A,\psi_A)}$ is isomorphic to $\mathrm{Sym}_B(\Omega^1_{B/A})$, where $\Omega^1_{B/A}$ is the module of Kähler differentials, cf. subsection 1.1, and we recover* (1.2) *from* (3.22)*.*

*The bijections* (3.21) *and* (3.22) *appear as the middle horizontal arrows in the diagram in remark 1.20 (2).*

*Let $(A, \delta_A)$ be a commutative differential ring that is a $k$-algebra and let $D := k[\mathbb{G}_a]$ be the coordinate ring of the additive group scheme $\mathbb{G}_a$. Then $A$ is a $D$-module algebra with respect to $\psi_A\colon D \otimes_k A \to A$ defined by $\psi_A(\partial^i \otimes a) := \delta^i_A(a)$ for all $a \in A$ and all $i \in \mathbb{N}$. As noted in remark 3.8 (2), the symmetric monoidal category of differential modules over $(A, \delta_A)$ is isomorphic to the symmetric monoidal category of left $(A\#D)$-modules over $A$ with respect to the product defined in* (3.18) *and commutative monoids in the category of left $(A\#D)$-modules are commutative differential $(A, \delta_A)$-algebras. Therefore we recognize proposition 1.24 as a corollary of proposition 3.9.*

3.3.2. *Endomorphisms.*

**Example 3.13.** *Let $D := k\langle 1, \sigma\rangle$ be the free $k$-module generated by two elements $1$ and $\sigma$ with the $k$-coalgebra structure defined by $\Delta(1) := 1 \otimes 1$, $\varepsilon(1) = 1$, $\Delta(\sigma) := \sigma \otimes \sigma$ and $\varepsilon(\sigma) = 1$. A unital $D$-measuring $\psi_A\colon D \otimes_k A \to A$ from $A$ to itself is the same as an endomorphism $\sigma_A$ of the $k$-algebra $A$ defined by $\sigma_A(a) := \psi_A(\sigma \otimes a)$. The $B$-algebra $\Omega^{D,1}_{B/(A,\psi_A)}$ has the form*

$$\Omega^{D,1}_{B/(A,\psi_A)} = A[1(b), \sigma(b) \mid b \in B]/I,$$

*where $I$ is the ideal generated by the elements*

$$\begin{array}{lll} \sigma(b+b') - \sigma(b) - \sigma(b'), & \sigma(bb') - \sigma(b)\sigma(b'), & \sigma(f(a)) - \psi_A(\sigma \otimes a) \\ 1(b+b') - 1(b) - 1(b'), & 1(bb') - 1(b)1(b'), & 1(f(a)) - a \end{array}$$

*for all $b, b' \in B$ and all $a \in A$. It is isomorphic to*

$$(B \otimes_{\mathbb{Z}} B)/J,$$

*where $J$ is the ideal generated by the elements $1 \otimes f(a) - \sigma_A(a) \otimes 1$ for all $a \in A$, via the isomorphism*

$$(B \otimes_{\mathbb{Z}} B)/J \to \Omega^{D}_{B/(A,\psi_A)}, \qquad \overline{b_1 \otimes b_2} \mapsto b_1\sigma(b_2).$$

We generalize the previous example:

**Example 3.14.** *For $n \in \mathbb{N}$, let $D_n := k\langle \sigma^0, \sigma^1 \dots \sigma^n\rangle$ be the free $k$-module with basis $\{\sigma^0, \sigma^1, \dots, \sigma^n\}$ with the $k$-coalgebra structure defined by*

$$\Delta(\sigma^i) := \sigma^i \otimes \sigma^i \quad \text{and} \quad \varepsilon(\sigma^i) = 1 \tag{3.23}$$

*for all $i = 0, \dots, n$. We denote the element $\sigma^0 \in D_n$ also by $1$. Let $\psi_n\colon D_n \otimes_k A \to A$ be a $D_n$-measuring from $A$ to itself and $(\tilde{\sigma}^i)_{i=0,\dots,n}$ be the family of endomorphisms $\tilde{\sigma}^i\colon A \to A$ defined by $\tilde{\sigma}^i(a) := \psi_n(\sigma^i \otimes a)$ for $i = 0, \dots, n$. The $B$-algebra $\Omega^{D_n,1}_{B/(A,\psi_n)}$ has the form*

$$\Omega^{D_n,1}_{B/(A,\psi_n)} = A[\sigma^0(b), \sigma^1(b), \dots, \sigma^n(b) \mid b \in B]/I,$$

*where $I$ is the ideal generated by the elements*

$$\sigma^i(b+b') - \sigma^i(b) - \sigma^i(b'), \quad \sigma^i(bb') - \sigma^i(b)\sigma^i(b') \quad \text{and} \quad \sigma^i(f(a)) - \psi_n(\sigma^i \otimes a).$$

*for all $b, b' \in B$, all $a \in A$ and all $i \in \{0, \dots, n\}$.*

*In the proof of proposition 1.32 rings $B_n$ are defined as*

$$B_n := B \otimes_A {}^{\tilde{\sigma}}B \otimes_A \cdots \otimes_A {}^{\widetilde{\sigma^n}}B,$$

*cf.* (1.18)*. There is an isomorphism of $A$-algebras*

$$B_n \xrightarrow{\sim} \Omega^{D_n,1}_{B/(A,\psi_n)}, \quad (b_0 \otimes a_0) \otimes (b_1 \otimes a_1) \otimes \cdots \otimes (b_n \otimes a_n) \mapsto \sigma^0(b_0) \cdot a_0 \cdot \sigma^1(b_1) \cdot a_1 \cdot \cdots \cdot \sigma^n(b_n) \cdot a_n \tag{3.24}$$

*with inverse given by*

$$\Omega^{D_n,1}_{B/(A,\psi_n)} \to B_n, \quad \sigma^i(b) \mapsto (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1) \otimes (b \otimes 1) \otimes (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1),$$

*for all $i = 0, \dots, n$ and $b \in B$, where $b \otimes 1$ is in the factor ${}^{\widetilde{\sigma^i}}B$ of $B_n$.*

*The $k$-bialgebra $D = k[\sigma]$ with $k$-coalgebra structure defined by* (3.23) *for all $i \in \mathbb{N}$ is the direct limit of the $k$-subcoalgebras $(D_i)_{i\in\mathbb{N}}$, which fulfill $D_iD_j \subseteq D_{i+j}$ for all $i, j \in \mathbb{N}$. Let $(A, \sigma_A)$ be a commutative difference ring and $\psi_A\colon D \otimes_k A \to A$, $\sigma^i \otimes a \mapsto \sigma^i_A(a)$ be the associated $D$-module algebra structure on $A$, cf. example 2.5 (3). As noted in remark 3.8 (4), the symmetric monoidal category of difference modules over $(A, \sigma_A)$ is isomorphic to the symmetric monoidal category of left $(A\#D)$-modules over $A$. Therefore the category of commutative difference $(A, \sigma_A)$-algebras is the isomorphic to the category of commutative monoids in the category of left $(A\#D)$-modules over $A$. The isomorphisms* (3.24) *induce an isomorphism of difference $(A, \sigma_A)$-algebras between $\Omega^{D,1}_{B/(A,\psi_A)}$ and the difference $(A, \sigma_A)$-algebra $[\sigma]_AB$ defined in proposition 1.32 and we recognize this proposition as a corollary of proposition 3.9.*

3.3.3. *Skew-derivations.*

**Example 3.15.** *Let $D$ be the free $k$-module $k\langle 1, \partial, \sigma\rangle$ with basis $\{1, \partial, \sigma\}$ and $k$-coalgebra structure defined by*

$$\Delta(1) \coloneqq 1 \otimes 1, \quad \varepsilon(1) \coloneqq 1, \quad \Delta(\sigma) \coloneqq \sigma \otimes \sigma, \quad \varepsilon(\sigma) \coloneqq 1, \quad \Delta(\partial) \coloneqq \partial \otimes 1 + \sigma \otimes \partial, \quad \varepsilon(\partial) \coloneqq 0. \tag{3.25}$$

*Then defining a $D$-measuring from $A$ to itself amounts to give an endomorphism $\sigma_A$ of the $k$-algebra $A$ and a $\sigma_A$-derivation $\partial_A$ on $A$, i.e. a morphism of $k$-modules $\partial_A\colon A \to A$ such that $\partial_A(aa') = \partial_A(a)a' + \sigma_A(a)\partial_A(a')$ for all $a, a' \in A$.*

The approach to skew-derivation taken by André in [And01] is different. Instead of working with genuine skew-derivations, he considers modules over a commutative difference ring $(A, \sigma_A)$ as sesqui-modules with respect to $\sigma_A$, equipped with a normal derivation.

3.3.4. *Iterative $q$-difference operators.* Hardouin introduced iterative $q$-difference operators in [Har10]. We do not give a detailed description of the generalized differentials in this case but only mention that commutative rings with iterative $q$-difference operators can be described as commutative $D$-module algebras for a cocommutative bialgebra $D$ as shown by Masuoka and Yanagawa, cf. [MY13]. Therefore our construction and propositions apply also in this case.

## 3.4. **Functorial properties.**

3.4.1. Let $v\colon B \to B'$ be a morphism of $A$-algebras. Then the morphism of $A$-algebras

$$A[db \mid d \in D, b \in B] \to A[db' \mid d \in D, b' \in B'], \quad ad(b) \mapsto ad(v(b))$$

induces a morphism of $A$-algebras

$$\Omega^D_{B/(A,\psi_A)} \to \Omega^D_{B'/(A,\psi_A)}. \tag{3.26}$$

If $D$ contains a group-like element 1 and $\psi_A(1 \otimes a) = a$ for all $a \in A$, then (3.26) is a morphism of $B$-algebras

$$\Omega^{D,1}_{B/(A,\psi_A)} \to \Omega^{D,1}_{B'/(A,\psi_A)},$$

where the $B$-algebra structure on $\Omega^{D,1}_{B'/(A,\psi_A)}$ is obtained from its $B'$-algebra structure and $v\colon B \to B'$. The diagram

$$\begin{tikzcd}
{}_k\mathcal{M}(D, \Omega^D_{B/(A,\psi_A)}) \arrow[r] & {}_k\mathcal{M}(D, \Omega^D_{B'/(A,\psi_A)}) \\
B \arrow[u, "\rho_u"] \arrow[r, "v"] & B' \arrow[u, "\rho_u"']
\end{tikzcd}$$

commutes, where the upper horizontal arrow is induced by (3.26). These morphisms induce a morphism of $B'$-algebras

$$v_{B'/B/A}\colon \Omega^{D,1}_{B/(A,\psi_A)} \otimes_B B' \to \Omega^{D,1}_{B'/(A,\psi_A)}.$$

3.4.2. Let $u\colon A \to A'$ be a morphism of commutative $k$-algebras and $f'\colon A' \to B'$ be a commutative $A'$-algebra. Let $\psi_A$ be a $D$-measuring from $A$ to itself and $\psi_{A'}$ be a $D$-measuring from $A'$ to itself extending $\psi_A$, i.e. such that the diagram

$$\begin{tikzcd}
D \otimes_k A' \arrow[r, "\psi_{A'}"] & A' \\
D \otimes_k A \arrow[u, "\mathrm{id}_D \otimes u"] \arrow[r, "\psi_A"] & A \arrow[u, "u"']
\end{tikzcd}$$

commutes.

We consider the morphism of $A$-algebras $A[db \mid d \in D, b \in B] \to A'[db \mid d \in D, b \in B]$ that is induced by $u$. It sends the element $d(f'(u(a))) - \psi_A(d \otimes a)$ to $d(f'(u(a))) - u(\psi_A(d \otimes a)) = d(f'(u(a))) - \psi_{A'}(d \otimes u(a))$, which vanishes in $\Omega^D_{B/(A,\psi_A)}$. Therefore we obtain a morphism of $A$-algebras

$$u_{B/A'/A}\colon \Omega^D_{B/(A,\psi_A)} \to \Omega^D_{B/(A',\psi_{A'})}.$$

3.4.3. Given a diagram of commutative $k$-algebras

$$\begin{tikzcd}
B \arrow[r, "v"] & B' \\
A \arrow[u, "f"] \arrow[r, "u"] & A' \arrow[u, "f'"']
\end{tikzcd}$$

and two $D$-measurings

$$\psi_A\colon D \otimes_k A \to A \qquad \text{and} \qquad \psi_{A'}\colon D \otimes_k A' \to A',$$

that are unital with respect to a group-like element $1 \in D$, the diagram

$$\begin{tikzcd}
{}_k\mathcal{M}(D, \Omega^{D,1}_{B/(A,\psi_A)} \otimes_B B') \arrow[r, "{}_k\mathcal{M}(D, v_{B'/B/A})"] & {}_k\mathcal{M}(D, \Omega^{D,1}_{B'/(A,\psi_A)}) \arrow[r, "{}_k\mathcal{M}(D, u_{B'/A'/A})"] & {}_k\mathcal{M}(D, \Omega^{D,1}_{B'/(A',\psi_{A'})}) \\
{}_k\mathcal{M}(D, \Omega^{D,1}_{B/(A,\psi_A)}) \otimes_B B' \arrow[u] & & \\
B \arrow[u, "\rho_u \otimes 1"] \arrow[r, "v"] & B' \arrow[uu, "\rho_u"] \arrow[r, "\mathrm{id}_{B'}"] & B' \arrow[uu, "\rho_u"]
\end{tikzcd}$$

commutes.

These are analogues of statements in [Gro64, Chapitre 0, 20.5].

## 4. REVIEW OF PROLONGATION SPACES

In this section we review the tangent bundle as well as several constructions of jet and prolongation spaces due to Buium, Rosen and Vojta.

**Notation:** *Let $A$ be a commutative ring and $f\colon A \to B$ be a commutative $A$-algebra.*

4.1. **The tangent bundle.** The *tangent bundle* of $X = \operatorname{Spec} B$ over $Y = \operatorname{Spec} A$ is by definition the $B$-scheme

$$\mathrm{T}_{X/Y} \coloneqq \operatorname{Spec} \operatorname{Sym}_B(\Omega^1_{B/A}),$$

where $\Omega^1_{B/A}$ is the module of Kähler differentials as defined in subsection 1.1, cf. [Gro67, §16.5]. We denote $\mathrm{T}_{X/Y}$ also by $\mathrm{T}_{B/A}$.

4.2. **Relative tangent bundle.** Given a derivation $\delta_A\colon A \to A$, we define the *relative tangent bundle*[7] of $X = \operatorname{Spec} B$ over $Y = \operatorname{Spec} A$ with respect to $\delta_A$ as

$$\mathrm{T}_{X/(Y,\delta_A)} \coloneqq \operatorname{Spec} \operatorname{Sym}_B(\Omega^1_{B/\mathbb{Z}})/I_{\delta_A},$$

where $I_{\delta_A}$ is the ideal of $\operatorname{Sym}_B(\Omega^1_{B/\mathbb{Z}})$ generated by $\mathrm{d}\, f(a) - f(\delta_A(a))$ for all $a \in A$ as in subsection 1.2.

---

[7] This is maybe not a standard notation.

4.3. **Vojta's scheme of jet differentials.** Vojta defines "schemes of jet differentials" for arbitrary schemes in [Voj07, §4]. Here we review his construction in the case of affine schemes.

**Definition 4.1.** *The* scheme of $m$-jet differentials *of* $\operatorname{Spec} B$ *over* $\operatorname{Spec} A$ *is defined as the* $B$*-scheme*

$$J_m(B/A) := \operatorname{Spec} \mathrm{HS}^m_{B/A},$$

*where* $\mathrm{HS}^m_{B/A}$ *is the* $B$*-algebra of divided differentials (cf. definition 1.7).*

**Theorem 4.2** ([IK03, 2.8], [Voj07, Theorem 4.5])**.** *The scheme* $J_m(B/A)$ *represents the functor from* $A$*-schemes to sets defined by*

$$Z \mapsto \mathsf{Sch}_A(Z \times_A A[t]/(t^{m+1}), \operatorname{Spec} B)$$

*in the case* $m \in \mathbb{N}$ *and by*

$$Z \mapsto \mathsf{Sch}_A(Z \hat{\times}_A A[\![t]\!]), \operatorname{Spec} B)$$

*in the case* $m = \infty$*, i.e. for every* $A$*-scheme* $Z$ *there are isomorphisms*

$$\mathsf{Sch}_A(Z \times_A A[t]/(t^{m+1}), \operatorname{Spec} B) \cong \mathsf{Sch}_A(Z, J_m(B/A)) \tag{4.1}$$

*and*

$$\mathsf{Sch}_A(Z \hat{\times}_A A[\![t]\!]), \operatorname{Spec} B) \cong \mathsf{Sch}_A(Z, J_\infty(B/A)),$$

*respectively.*

**Remark 4.3.**
1. *Ishii and Kollar call* $J_m(B/A)$ *the* scheme of $m$-jets *of* $X = \operatorname{Spec} B$ *and* $J_\infty(B/A)$ *the* space of arcs *of* $X$*.*
2. *By remark 1.8 the* $B$*-scheme* $J_1(B/A)$ *is isomorphic to the tangent bundle* $\mathrm{T}_{B/A}$ *as defined in subsection 4.1.*

4.4. **Buium's jet spaces.** Buium defines jet spaces in [Bui93, 9. Appendix]. We briefly recall his definition.

Let $A$ be a commutative $\mathbb{Q}$-algebra and $\delta_A \colon A \to A$ be a derivation. We define $A_m := A[t]/(t^{m+1})$, $Y := \operatorname{Spec} A$ and $Y^{(m)} := \operatorname{Spec} A[t]/(t^{m+1})$. We denote by $p_1 \colon Y^{(m)} \to Y$ the morphism induced by the inclusion

$$A \to A_m, \quad a \mapsto a.$$

and by $p_2 \colon Y^{(m)} \to Y$ the morphism induced by the homomorphism

$$e \colon A \to A_m, \quad a \mapsto \sum_{i=0}^{m} \frac{\delta_A^i(a)}{i!} t^i. \tag{4.2}$$

Given an $A$-scheme $X$, we consider the functor

$$\mathcal{G} \colon \mathsf{Sch}_A \to \mathsf{Set}, \quad Z \mapsto \mathsf{Sch}_A(Z \times_Y Y^{(m)}, X),$$

where $Y^{(m)}$ in the fibre product is considered as $Y$-scheme via $p_1$ and $Z \times_Y Y^{(m)}$ is considered as $Y$-scheme via the composition $Z \times_Y Y^{(m)} \to Y^{(m)} \xrightarrow{p_2} Y$. The functor $\mathcal{G}$ is representable, i.e. there exists an $A$-scheme $jet_m(X/Y, \delta_A)$ such that for every $A$-scheme $Z$ there is a bijection

$$\mathsf{Sch}_A(Z, jet_m(X/Y, \delta_A)) \cong \mathsf{Sch}_A(Z \times_Y Y^{(m)}, X). \tag{4.3}$$

**Remark 4.4.** *If the derivation* $\delta_A$ *on* $A$ *is trivial, i.e. if* $\delta_A(a) = 0$ *for all* $a \in A$*, then Buium's jet spaces* $jet_m(X/Y, \delta_A)$ *coincides with Vojta's scheme of* $m$*-jet differentials* $J_m(B/A)$*.*

4.5. **Rosen's prolongation spaces.** Rosen defines prolongation spaces of schemes in [Ros08]. We recall his definition in the affine case.

**Definition 4.5.** *Let* $A$ *be a commutative ring with higher derivation* $\delta_A = (\delta_A^{(i)})_{i \in \mathbb{N}}$ *and let* $X = \operatorname{Spec} B$ *be an affine* $A$*-scheme. The* $m$-th prolongation *of* $X$ *is defined as the* $B$*-scheme*

$$P_m(X/(A, \delta_A)) := \operatorname{Spec} \mathrm{HS}^m_{B/(A,\delta_A)}.$$

*The* $B$*-schemes* $(P_m(X, (A, \delta_A)))_{m \in \mathbb{N}}$ *form an inverse system and we denote the inverse limit by*

$$P_\infty(X/(A, \delta_A)) := \varprojlim_{m \in \mathbb{N}} P_m(X/(A, \delta_A)).$$

**Proposition 4.6** ([Ros08, Theorem 2.4])**.** *Let $A$ be a commutative ring with higher derivation $\delta_A = (\delta_A^{(i)})_{i\in\mathbb{N}}$ and let $X = \operatorname{Spec} B$ be an affine $A$-scheme. For all $m \in \mathbb{N}$ the scheme $P_m(X/(A,\delta_A))$ represents the functor*

$$\begin{aligned} \mathsf{Sch}_A &\to \mathsf{Set} \\ Z &\mapsto \mathsf{Sch}_A((Z \times_A \operatorname{Spec} A_m)^{\sim}, X), \end{aligned}$$

*where $(Z \times_A \operatorname{Spec} A_m)^{\sim}$ is an $A$-scheme via*

$$(Z \times_A \operatorname{Spec} A_m)^{\sim} \to \operatorname{Spec} A_m \xrightarrow{\underline{e}} \operatorname{Spec} A,$$

*where $\underline{e}\colon \operatorname{Spec} A_m \to \operatorname{Spec} A$ is the morphism induced by the ring homomorphism*

$$e\colon A \to A_m, \quad a \mapsto \sum_{i=0}^{m} \delta_A^{(i)}(a) t^i,$$

*i.e. for every $A$-scheme $Z$ there is a bijection*

$$\mathsf{Sch}_A(Z, P_m(X/(A,\delta_A))) \cong \mathsf{Sch}_A((Z \times_A \operatorname{Spec} A_m)^{\sim}, X). \tag{4.4}$$

*Proof.* Let $Z = \operatorname{Spec} R$ be an affine $A$-scheme. Then using (1.12) we have

$$\begin{aligned} \mathsf{Sch}_A(Z, P_m(X/(A,\delta_A))) &= P_m(X/(A,\delta_A))(R) \\ &= (\operatorname{Spec} \mathrm{HS}^m_{B/(A,\delta_A)})(R) \\ &= \mathsf{Alg}_A(\mathrm{HS}^m_{B/(A,\delta_A)}, R) \\ &\cong \mathsf{Alg}_A(B, \tilde{R}_m) \\ &\cong \mathsf{Alg}_A(B, R \otimes_A \tilde{A}_m) \\ &= \mathsf{Sch}_A((Z \times_A \operatorname{Spec} A_m)^{\sim}, X). \end{aligned}$$

□

**Remark 4.7.**

1. *If $A$ is a commutative $\mathbb{Q}$-algebra, $\delta_A^{(1)}\colon A \to A$ is a derivation and $\delta_A := (\frac{(\delta_A^{(1)})^i}{i!})_{i\in\mathbb{N}}$ is the induced higher derivation on $A$, then the prolongation space $P_m(X/(A,\delta_A))$ as defined by Rosen coincides with the jet schemes $jet_m(X/Y, \delta_A^{(1)})$ defined by Buium, where $Y := \operatorname{Spec} A$. The defining isomorphism* (4.3) *corresponds to* (4.4)*.*
2. *If $m = 1$ and if $\delta_A = (\delta_A^{(0)}, \delta_A^{(1)})$ is given by $\delta_A^{(0)} = \mathrm{id}_A$ and a derivation $\delta_A^{(1)}$ on $A$, then by remark 1.16 (2) the prolongation space $P_1(X/(A,\delta_A)) \cong jet_1(X/Y, \delta_A^{(1)})$ is isomorphic to the relative tangent bundle $\mathrm{T}_{X/(Y,\delta_A^{(1)})}$ as defined in subsection 4.2, where $Y := \operatorname{Spec} A$.*

**Example 4.8.** *Let $m = 1$, $\delta_A = (\mathrm{id}_A, \delta_A^{(1)})$, $X = \operatorname{Spec} A[x_1,\dots,x_n]/(Q_1,\dots,Q_s)$ with polynomials $Q_i \in A[x_1,\dots,x_n]$ and $R$ be a commutative $A$-algebra. Then we have*

$$P_1(X/(A,\delta_A))(R) \cong \mathsf{Alg}_A(A[x_1,\dots,x_n]/(Q_1,\dots,Q_s), \tilde{R}_1). \tag{4.5}$$

*A homomorphism of $A$-algebras from $A[x_1,\dots,x_n]/(Q_1,\dots,Q_s)$ to $\tilde{R}_1$ is given by the images $r_1^{(0)} + r_1^{(1)}t, \dots, r_n^{(0)} + r_n^{(1)}t$ of $x_1,\dots,x_n$ in $\tilde{R}_1$ such that the polynomials $Q_i$ vanish on them. This latter condition means (remember that the $A$-algebra structure on $\tilde{R}_1$ is induced by the truncated Taylor homomorphism $e\colon A \to A_1$) that*[8]

$$\begin{aligned} 0 &= Q_i^e(r_1^{(0)} + r_1^{(1)}t, \dots, r_n^{(0)} + r_n^{(1)}t) \\ &= Q_i(r_1^{(0)},\dots,r_n^{(0)}) + \sum_{j=1}^{n} \frac{\partial Q_i}{\partial x_j}(r_1^{(0)},\dots,r_n^{(0)}) r_j^{(1)} t + Q_i^{\delta_A^{(i)}}(r_1^{(0)},\dots,r_n^{(0)}) t, \end{aligned}$$

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[8] By $Q_i^e$ and by $Q_i^{\delta_A^{(1)}}$ we denote the polynomials obtained from $Q_i$ by applying $e$ and $\delta_A^{(1)}$ to the coefficients, respectively.

*where* $Q_i^e$ *denotes the image of* $Q_i$ *under the homomorphism* $A[x_1,\dots,x_n] \to A_1[x_1,\dots,x_n]$ *induced by* $e$. *Therefore the* $R$*-points of* $P_1(X/(A,\delta_A))$ *are given by the algebraic subset of* $\mathbb{A}^{2n}(R)$ *consisting in the points* $(r_1^{(0)},\dots,r_n^{(0)},r_1^{(1)},\dots,r_n^{(1)})$ *fulfilling the polynomial equations*

$$Q_i(r_1^{(0)},\dots,r_n^{(0)})=0, \qquad \sum_{j=1}^{n}\frac{\partial Q_i}{\partial x_j}(r_1^{(0)},\dots,r_n^{(0)})r_j^{(1)}+Q_i^{\delta_A^{(1)}}(r_1^{(0)},\dots,r_n^{(0)})=0$$

*for all* $i=1,\dots,s$.

## 5. Generalized prolongation spaces

In section 4 of [MS10], Moosa and Scanlon define general prolongation spaces in terms of Weil restrictions and show their existence in important cases. Here we give a direct construction in the case of affine schemes, which seems to be more direct and is analogous to the constructions of Buium, Rosen and Vojta that we reviewed in section 4. We obtain the prolongation spaces as the spectra of the $A$-algebras $\Omega^D_{B/(A,\psi_A)}$ defined in section 3.

For the convenience of the reader we first recall the necessary definitions leading to the notion of prolongation spaces in the sense of Moosa and Scanlon.

**Notation:** *We denote by* $\mathbb{S}$ *the* standard ring scheme *over* $k$, *i.e. the* $k$*-scheme* $\operatorname{Spec} k[x]$ *regarded as a ring scheme by equipping for every commutative* $k$*-algebra* $A$ *the set* $\mathbb{S}(A)\cong A$ *with the given ring structure of* $A$.

**Definition 5.1** ([MS10, Definition 3.1] and [MS11, Definition 2.1])**.** *A* finite free commutative $\mathbb{S}$-algebra scheme *is an affine commutative* $\mathbb{S}$*-algebra scheme* $\mathcal{E}$ *that is isomorphic to* $\mathbb{S}^l$ *as* $\mathbb{S}$*-module scheme for some* $l\in\mathbb{N}$.

**Definition 5.2** ([MS10, Definition 3.3])**.** *Given a finite free commutative* $\mathbb{S}$*-algebra scheme* $\mathcal{E}$, *a* commutative $\mathcal{E}$-ring (over $k$) *is a commutative* $k$*-algebra* $A$ *together with a homomorphism* $e\colon A\to\mathcal{E}(A)$ *of* $k$*-algebras.*

**Notation:** *If* $\mathcal{E}$ *is a finite free* $\mathbb{S}$*-algebra scheme and* $e\colon A\to\mathcal{E}(A)$ *is a commutative* $\mathcal{E}$*-ring, then we denote the ring* $\mathcal{E}(A)$, *considered as* $A$*-algebra via the homomorphism* $e\colon A\to\mathcal{E}(A)$, *by* $\mathcal{E}^e(A)$. *By* $\mathcal{E}(A)$ *we denote the same ring, but with the* $A$*-algebra structure* $A=\mathbb{S}(A)\to\mathcal{E}(A)$ *induced from the* $\mathbb{S}$*-algebra structure on* $\mathcal{E}$.

**Definition 5.3** ([MS10, Definition 4.1])**.** *Let* $\mathcal{E}$ *be a finite free commutative* $\mathbb{S}$*-algebra scheme over* $k$, $e\colon A\to\mathcal{E}(A)$ *be a commutative* $\mathcal{E}$*-ring and* $X$ *be an* $A$*-scheme. The* prolongation space of $X$ with respect to $\mathcal{E}$ and $e$, *denoted by* $\tau(X,\mathcal{E},e)$, *is the Weil restriction of* $X\times_A\mathcal{E}^e(A)$, *where we consider* $\mathcal{E}(A)$ *as* $A$*-algebra via* $e$ *to form the base extension, from* $\mathcal{E}(A)$ *to* $A$ *via the standard* $A$*-algebra structure on* $\mathcal{E}(A)$, *if it exists.*

**Remark 5.4.** *(1) We recall the Weil restriction: Let* $A$ *be a commutative ring,* $B$ *be a commutative* $A$*-algebra that is finite and free over* $A$, *and let* $W$ *be a scheme over* $B$ *such that the morphism* $\operatorname{Spec}B\to\operatorname{Spec}A$ *is a homeomorphism or* $W$ *has the property that every finite set of points is contained in an affine open subset. Then there exists an* $A$*-scheme* $\mathrm{R}_{B/A}W$ *such that for every* $A$*-scheme* $Z$ *there is a bijection*

$$\mathsf{Sch}_B(Z\times_A B,W)\cong\mathsf{Sch}_A(Z,\mathrm{R}_{B/A}W).$$

*The* $A$*-scheme* $\mathrm{R}_{B/A}W$ *is called the* Weil restriction *of* $W$ *from* $B$ *to* $A$. *For details we refer to* [MS10, Theorem 2.1] *or* [BLR90, Section 7.6, Theorem 4].

*(2) Let* $\mathcal{E}$ *be a finite free commutative* $\mathbb{S}$*-algebra scheme over* $k$, *let* $e\colon A\to\mathcal{E}(A)$ *be a commutative* $\mathcal{E}$*-ring, and* $X$ *be an* $A$*-scheme. There is a canonical bijection*

$$\mathsf{Sch}_A(Z\times_A\mathcal{E}(A),X)\cong\mathsf{Sch}_{\mathcal{E}(A)}(Z\times_A\mathcal{E}(A),X\times_A\mathcal{E}^e(A)), \tag{5.1}$$

*where* $Z\times_A\mathcal{E}(A)$ *on the left hand side is an* $A$*-scheme via*

$$Z\times_A\mathcal{E}(A)\to\operatorname{Spec}\mathcal{E}(A)\xrightarrow{\underline{e}}\operatorname{Spec}A, \tag{5.2}$$

*where* $\underline{e}\colon\operatorname{Spec}\mathcal{E}(A)\to\operatorname{Spec}A$ *is the morphism induced by the ring homomorphism* $e\colon A\to\mathcal{E}(A)$.

*(3) Therefore, for the prolongation space* $\tau(X,\mathcal{E},e)$ *we have bijections*

$$\mathsf{Sch}_A(Z\times_A\mathcal{E}(A),X)\cong\mathsf{Sch}_{\mathcal{E}(A)}(Z\times_A\mathcal{E}(A),X\times_A\mathcal{E}^e(A))\cong\mathsf{Sch}_A(Z,\tau(X,\mathcal{E},e)), \tag{5.3}$$

*where again* $Z\times_A\mathcal{E}(A)$ *on the left hand side is an* $A$*-scheme via* (5.2), *and the right factor in* $X\times_A\mathcal{E}^e(A)$ *is an* $A$*-algebra via* $e$.

If $\mathcal{E}$ is a finite free commutative $\mathbb{S}$-algebra scheme over $k$, then the dual $D := \mathcal{E}(k)^* := {}_k\mathcal{M}(\mathcal{E}(k), k)$ of the commutative $k$-algebra $\mathcal{E}(k)$ is a cocommutative $k$-coalgebra. There are isomorphisms of $k$-algebras

$$\mathcal{E}(A) \cong \mathcal{E}(k) \otimes_k A \cong D^* \otimes_k A \cong {}_k\mathcal{M}(D, A) \tag{5.4}$$

and an $\mathcal{E}$-ring structure $e\colon A \to \mathcal{E}(A)$ on a commutative $k$-algebra $A$ induces a homomorphism $\rho_A\colon A \to {}_k\mathcal{M}(D, A)$ of $k$-algebras and thus a $D$-measuring $\psi_A\colon D \otimes_k A \to A$ from $A$ to itself, cf. section 2. For further details on this duality we refer to [Hei13b].

**Remark 5.5.** *The homomorphism* $e\colon A \to \mathcal{E}(A)$ *generalizes the homomorphism* $e\colon A \to A[t]/(t^{m+1})$ *in* (1.10) *and therefore the* $A$*-algebra* $\mathcal{E}^e(A)$ *generalizes the* $A$*-algebra* $\tilde{A}_m$ *in definition 1.18.*

**Proposition 5.6.** *Let* $\mathcal{E}$ *be a finite free commutative* $\mathbb{S}$*-algebra scheme. We define* $D := \mathcal{E}(k)^*$ *to be the cocommutative* $k$*-coalgebra associated to* $\mathcal{E}$*. Let* $e\colon A \to \mathcal{E}(A)$ *be a commutative* $\mathcal{E}$*-ring,* $\rho_A\colon A \to {}_k\mathcal{M}(D, A)$ *be the composition of* $e$ *with* (5.4) *and let* $\psi_A\colon D \otimes_k A \to A$ *be the associated* $D$*-measuring. Let* $f\colon A \to B$ *be a commutative* $A$*-algebra and* $X := \operatorname{Spec} B$*. Then the* $A$*-scheme* $\operatorname{Spec} \Omega^D_{B/(A,\psi_A)}$ *is isomorphic to the prolongation space* $\tau(X, \mathcal{E}, e)$ *of* $X$ *with respect to* $\mathcal{E}$ *and* $e$*, i.e. for every affine* $A$*-scheme* $Z = \operatorname{Spec} R$ *there is an isomorphism*

$$\mathsf{Sch}_A(Z, \operatorname{Spec} \Omega^D_{B/(A,\psi_A)}) \cong \mathsf{Sch}_{\mathcal{E}(A)}(Z \times_A \mathcal{E}(A), X \times_A \mathcal{E}^e(A)) \cong \mathsf{Sch}_A(Z \times_A \mathcal{E}(A), X), \tag{5.5}$$

*where the* $A$*-scheme structure on* $Z \times_A \mathcal{E}(A)$ *is given by* $Z \times_A \mathcal{E}(A) \to \operatorname{Spec} \mathcal{E}(A) \xrightarrow{e} \operatorname{Spec} A$*.*

*Proof.* We denote by $h\colon A \to R$ the $A$-algebra structure of $R$. The isomorphisms (5.4) induce isomorphisms of $A$-algebras

$$R \otimes_A \mathcal{E}(A) \cong R \otimes_A {}_k\mathcal{M}(D, A) \cong {}_k\mathcal{M}(D, R),$$

where in $R \otimes_A \mathcal{E}(A)$ and $R \otimes_A {}_k\mathcal{M}(D, A)$ the $A$-algebra structures on $\mathcal{E}(A)$ and ${}_k\mathcal{M}(D, A)$ are the canonical ones to form the tensor product. We consider $R \otimes_A {}_k\mathcal{M}(D, A)$ and ${}_k\mathcal{M}(D, R)$ as $A$-algebras via the homomorphisms $\rho_A\colon A \to {}_k\mathcal{M}(D, A)$ and ${}_k\mathcal{M}(D, h) \circ \rho_A$, respectively, and $\mathcal{E}(A)$ via $e\colon A \to \mathcal{E}(A)$. Therefore we obtain bijections[9]

$$\mathsf{Sch}_{\mathcal{E}(A)}(Z \times_A \mathcal{E}(A), X \times_A \mathcal{E}^e(A)) \cong \mathsf{Sch}_A(Z \times_A \mathcal{E}(A), X) \cong \mathsf{Alg}_A(B, R \otimes_A \mathcal{E}(A)) \cong \mathsf{Alg}_A(B, {}_k\mathcal{M}(D, R)),$$

where $R \otimes_A \mathcal{E}(A)$ and ${}_k\mathcal{M}(D, R)$ are considered as $A$-algebras via $e$ and ${}_k\mathcal{M}(D, h) \circ \rho_A$, respectively. At the other side, by proposition 3.2 (1) we have

$$\mathsf{Sch}_A(Z, \operatorname{Spec} \Omega^D_{B/(A,\psi_A)}) \cong \mathsf{Alg}_A(\Omega^D_{B/(A,\psi_A)}, R) \cong \mathsf{Alg}_A(B, {}_k\mathcal{M}(D, R)).$$

□

**Example 5.7** (Rings with higher derivation)**.** *(1) If* $\mathcal{E}$ *is the* $\mathbb{S}$*-algebra scheme defined by* $\mathcal{E}(A) := A[t]/(t^{m+1})$ *for some* $m \in \mathbb{N}$ *and every commutative* $k$*-algebra* $A$*, then the* $k$*-coalgebra* $D := \mathcal{E}(k)^*$ *associated to* $\mathcal{E}$ *is the free* $k$*-module* $D = k\langle \theta^{(0)}, \ldots, \theta^{(m)} \rangle$ *with comultiplication* $\Delta$ *and counit* $\varepsilon$ *given by the homomorphisms of* $k$*-modules defined by*

$$\Delta(\theta^{(i)}) = \sum_{i=i_1+i_2} \theta^{(i_1)} \otimes \theta^{(i_2)} \qquad \text{and} \qquad \varepsilon(\theta^{(i)}) = \delta_{i,0}$$

*for all* $i = 0, \ldots, m$ *as in example 2.5 (1). Commutative* $\mathcal{E}$*-rings* $e\colon A \to \mathcal{E}(A)$ *(over* $k$*) correspond to commutative* $k$*-algebras* $A$ *with* $D$*-measuring* $\psi_A\colon D \otimes_k A \to A$*, i.e. with Hasse-Schmidt derivation* $\delta_A = (\delta_A^{(i)})_{i=0,\ldots,m}$ *of length* $m$*, cf. example 3.11 and* [Hei13b, Example 8.1].

*Let* $f\colon A \to B$ *be a commutative* $A$*-algebra and* $X := \operatorname{Spec} B$*. Assume that* $\psi_A\colon D \otimes_k A \to A$ *is a* $D$*-measuring. Then the isomorphism between* $\operatorname{Spec} \Omega^D_{B/(A,\psi_A)}$ *and* $\tau(\operatorname{Spec} B, \mathcal{E}, e)$ *is the defining isomorphism*

$$P_m(X/(A, \delta_A)) \cong \operatorname{Spec} \mathrm{HS}^m_{B/(A,\delta_A)}$$

*of Rosen's prolongation space* $P_m(X/(A, \delta_A))$*. The isomorphism* (5.5) *specializes to* (4.4)*.*

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[9] In $Z \times_A \mathcal{E}(A)$ and $R \otimes_A \mathcal{E}(A)$, the ring $\mathcal{E}(A)$ is considered as $A$-algebra via the $\mathbb{S}$-algebra structure of $\mathcal{E}$ to form the base extension, but the product is made into an $A$-algebra via $A \xrightarrow{e} \mathcal{E}(A) \to R \otimes_A \mathcal{E}(A)$. The $A$-algebra structure of $\mathcal{E}(A)$ that is used to form the base extensions $X \times_A \mathcal{E}^e(A)$ is given by $e\colon A \to \mathcal{E}(A)$.

*(2) If in the situation of (1) the higher derivation* $(\delta_A^{(i)})_{i=0,\dots,m}$ *on* $A$ *is trivial, then* $\tau(\operatorname{Spec} B, \mathcal{E}, e)$ *is isomorphic to the scheme of* $m$*-jet differentials* $J_m(B/A) = \operatorname{Spec} \mathrm{HS}^m_{B/A}$ *as defined by Vojta (cf. definition 4.1) and the isomorphism* (5.5) *specializes to* (4.1)*.*

We illustrate the previous example in the case $m = 1$:

**Example 5.8** (Differential rings)**.** *Let* $\mathcal{E}$ *be the finite free commutative* $\mathbb{S}$*-algebra scheme defined by* $\mathcal{E}(A) = A[t]/(t^2)$ *for every commutative* $k$*-algebra* $A$*. Let further* $(A, \delta_A^{(1)})$ *be a differential field,*

$$e\colon A \to \mathcal{E}(A) = A[t]/(t^2), \quad a \mapsto a + \delta_A^{(1)}(a)t$$

*be the truncated Taylor morphism induced by* $\delta_A^{(1)}$ *and* $\delta_A = (\mathrm{id}_A, \delta_A^{(1)})$ *be the unital higher derivation of length* 1 *induced by* $\delta_A^{(1)}$*. Let* $B$ *be a commutative* $A$*-algebra,* $X \coloneqq \operatorname{Spec} B$ *and* $Y \coloneqq \operatorname{Spec} A$*. Example 5.7 shows that the prolongation space* $\tau(X, \mathcal{E}, e)$ *is isomorphic to the prolongation space* $P_1(X/(A, \delta_A))$ *as defined by Rosen, which is isomorphic to the relative tangent bundle* $T_{X/(Y,\delta_A^{(1)})}$ *by remark 4.7 (2).*

*If the derivation* $\delta_A^{(1)}$ *on* $A$ *is trivial, then* $P_1(X, (A, \delta_A))$ *coincides with* $J_1(B/A)$ *as defined by Vojta (cf. definition 4.1), which is isomorphic to the tangent bundle* $T_{X/Y}$ *by remark 4.3 (2).*

*Now we assume that* $B = A[x_1, \dots, x_n]/(Q_1, \dots, Q_s)$ *and let* $X = \operatorname{Spec}(B)$ *be the algebraic subvariety of* $\mathbb{A}^m = \operatorname{Spec} A[x_1, \dots x_n]$ *defined by the polynomials* $Q_1, \dots, Q_s \in A[x_1, \dots x_n]$ *over* $A$*. Then the prolongation space* $\tau(X, \mathcal{E}, e)$ *is given by the subspace of* $\mathbb{A}^{2m}$ *defined by*

$$\operatorname{Spec}(A[x_1, \dots, x_n, x_1^{(1)}, \dots, x_n^{(1)}]/\Big(\{Q_j, \sum_{i=1}^{n} \frac{\partial Q_j}{x_i} x_i^{(1)} + Q_j^{\delta_A^{(1)}}) \mid j = 1, \dots, s\}\Big),$$

*where* $Q_j^{\delta_A^{(1)}}$ *denotes the polynomials obtained from* $Q_j$ *by applying* $\delta_A^{(1)}$ *to each coefficient. This space coincides with the first prolongation space of Rosen, cf. example 4.8.*

*If* $X$ *is defined over the constants of* $A$*, i.e. if the coefficients of the polynomials* $Q_j$ *are constant with respect to the derivation* $\delta_A^{(1)}$*, then this specializes to*

$$\operatorname{Spec}(A[x_1, \dots, x_n, x_1^{(1)}, \dots, x_n^{(1)}]/(\{Q_j, \sum_{i=1}^{n} \frac{\partial Q_j}{x_i} x_i^{(1)} \mid j = 1, \dots, s\})),$$

*which is the tangent bundle of* $X$*, cf. subsection 4.1.*

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